UNIVERSIDAD NACIONAL DE COLOMBIA

Sobre el problema de restricción de la transformada de Fourier y sus aplicaciones

Felipe Eduardo Ponce Vanegas

Universidad Nacional de Colombia
Facultad de Ciencias, Departamento de Matemáticas
Bogotá, Colombia
Año 2017

Sobre el problema de restricción de la transformada de Fourier y sus aplicaciones

Felipe Eduardo Ponce Vanegas

Trabajo final presentado como requisito parcial para optar al título de:
Doctor en Matemáticas

Director:
Javier Ramos Maravall

Codirector:
Germán Eduardo Fonseca Buitrago

Línea de Investigación:
Análisis armónico

Universidad Nacional de Colombia
Facultad de Ciencias, Departamento de Matemáticas
Bogotá, Colombia
Año 2017

Agradecimientos

A mi Dios doy gracias por acompañarme siempre en mi vida y darme todo lo que tengo. Agradezco a mi familia, mis padres y mis hermanos, por haber constituido tan fuerte soporte en mi vida espiritual. Sin su amor y ayuda, no habría sido posible llegar a este punto de mi vida.

Expreso mi gratitud a Javier Ramos, mi director de tesis, quien compartió conmigo sus conocimientos y experiencia en análisis armónico con generosidad, sin negarme cualquier ayuda que él haya podido darme. Las discusiones que tuvimos acerca del área me ayudaron a ampliar mis conocimientos y me han provisto de una buena preparación para la vida académica. También quiero agradecer a Germán Fonseca, quien nunca se negó a ayudarme y pacientemente me llevó en todo este proceso.

Ana Vargas y Keith Rogers siempre me extendieron sus manos para apoyarme, y propiciaron para mí una oportunidad para ir al Congreso de El Escorial en España, un reconocido evento en mi área de investigación.

Siempre fue edificante y agradable compartir un momento con mis amigos, como Frank Trujillo, Oscar Riaño, Jorge Morales, Razvan Iagar y Diego Tapias. También fueron enriquecedores los seminarios con los profesores Germán Fonseca, Guillermo Rodríguez, Félix Soriano, Ricardo Pastrán y Omar Duque, además de mis compañeros de estudio. Me disculpo por todos aquellos colaboradores que no mencioné, pues siempre hubo muchos.

Esta tesis fue financiada por Colciencias, convocatorio 6172.

Title. On the Restriction Problem in Harmonic Analysis and its Applications

Abstract

Since the seventies, the interest in understanding the mapping properties of restricting the Fourier transform of a function to a manifold, has triggered important new lines of research in analysis. In this thesis we focus on the multilinear theory of restriction, in particular, we extend to the hyperbolic paraboloid a theorem of Ramos about elliptic surfaces in $\mathbb{R}^3$, who got the sharp dependence on transversality in the multilinear inequality of Bennett, Carbery and Tao. Furthermore, we point to a possible route towards the proof of Ramos' theorem in higher dimensions.

We show also an application of restriction theory to Falconer's conjecture, a problem in geometric measure theory. This problem relates to the rate of decay of spherical means of the Fourier transform of compactly supported measures. We exhibit measures whose Fourier transform decays slowly in the whole space, in contrast to previous results.

Keywords: Restriction operator, multilinear inequalities, transversality, Falconer's conjecture

Título. Sobre el problema de restricción de la transformada de Fourier y sus aplicaciones

Resumen

A partir de los años setenta surgió un especial interés en entender las propiedades del operador de restricción de la transformada de Fourier de una función a una variedad, lo que ha promovido nuevas líneas de investigación relevantes en análisis. En esta tesis nos enfocamos en la teoría de restricción multilineal, en particular, extendemos al paraboloide hiperbólico un teorema de Ramos sobre superficies elípticas en $\mathbb{R}^3$, quien obtuvo la dependencia óptima de la transversalidad en la desigualdad multilineal de Bennett, Carbery y Tao. Adicionalmente, señalamos una posible ruta hacia la demostración del teorema de Ramos en dimensiones mayores.

También mostramos una aplicación de la teoría de restricción a la conjetura de Falconer, un problema en teoría geométrica de la medida. Este problema se relaciona con la decaída del promedio esférico de la transformada de Fourier de medidas con soporte compacto. Construimos una única medida que decae lentamente en todo el espacio, en contraste con resultados anteriores.

Palabras clave: Operador de restricción, desigualdades multilineales, transversalidad, conjetura de Falconer.

Contents

List of Symbols

Symbol	Description
B_r	A ball of radius r.
S^{n-1}	Unitary sphere in $\mathbb{R}^n$.
$\mathcal{S}$	Schwartz space.
$\|E\|$	Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. If E is a discrete set, then it denotes its cardinality.
$\lesssim_\epsilon$ ($\gtrsim_\epsilon$)	Denotes the relation $\|A\| \leq C_\epsilon B$ ($\|A\| \geq C_\epsilon B$) for some constant C that depends on the parameter ϵ.
$A \sim B$	Denotes the relation $B \lesssim A \lesssim B$.
$\ll$ ($\gg$)	Denotes the relation $\|A/B\| \leq c$ ($\|A/B\| \geq C$) for some sufficiently small (large) constant c (C).
$A \approx B$	A and B are similar in some vague sense depending on the context.
$f = o(g)$	Denotes the relation $f(x)/g(x) \to 0$ if x approach some given number.
π_j	The projection $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$, sending x to $\pi_j x = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.
ι_j	The inclusion $\iota_j : \mathbb{R}^{n-1} \to \mathbb{R}^n$, sending x to $\iota_j x = (x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_{n-1})$.
$\langle \cdot \rangle$	Japanese bracket; $\langle \xi \rangle = (1 + \|\xi\|^2)^{1/2}$.
$e(\cdot)$	$e(x) = e^{2\pi i x}$.
t_+	Denotes $\max\{t, 0\}$.
$\mathbb{1}_E$	Indicator function of E, defined as $\mathbb{1}_E(x) = 1$ if $x \in E$ and $\mathbb{1}_E(x) = 0$ if $x \notin E$.
ζ_E	Bump function in $\mathcal{S}$ adapted to the set E. We do not require from ζ compact support, but localization to the set of interest.
p'	Conjugate exponent of p, defined as $1 = \frac{1}{p} + \frac{1}{p'}$.
$d(x, E)$	The distance from a point x to the set E, defined as $\inf\{\|x - y\| \mid y \in E \subset \mathbb{R}^n\}$.
$\Delta(E)$	The distance set $\{\|x - y\| \mid x, y \in E \subset \mathbb{R}^n\} \subset \mathbb{R}_{\geq 0}$.
$\mathcal{H}^s$	Spherical Hausdorff content of dimension s.
$\dim(E)$	Hausdorff dimension of the set E.
$I_t(\mu)$	Potential of the measure μ, defined as $\int \|x - y\|^{-t} \, d\mu(x) d\mu(y)$.
$\fint_B f$	The average of f over the ball B, defined as $\frac{1}{\|B\|} \int_B f$.
$\tilde{U}$	If $U \subset \mathbb{R}^{n-1}$ and S is the graph of a function φ in U, then $\tilde{U}$ is the lift of U to S, that is, $\tilde{U} = \{(x, \varphi(x)) \mid x \in U\}$.
$\tilde{a}$	If $a \in \mathbb{R}^{n-1}$ and S is the graph of a function φ, then $\tilde{a} = (a, \varphi(a))$.
$f d\sigma$	If f is a function in $U \subset \mathbb{R}^{n-1}$ and S the graph of a function φ in U, the lift of f to S is the measure $f(x') dV(x', x_n)/\sqrt{1 + \|\nabla \varphi(x')\|^2}$, where dV is the volume element of S.

1 Introduction

I wish to present here a brief overview of the development around the restriction conjecture, giving only sketchy proofs, and warning that many statements may require heavy technical work to get into a rigorous proof. Some important work will be omitted, in favor of topics more closely related to this thesis; for a more detailed and complete exposition, see *e.g.* [65, 67, 76, 1, 54].

A basic principle in harmonic analysis says that the faster the Fourier transform of a function decays, the smoother the function is. This principle has several realizations in different theorems and estimations in analysis, however our understanding is still unsatisfactory and many fundamental questions remain open. Among the many ways to measure the size of a function, the L^p norm is one of the most usual methods, hence it is natural to ask for the relationship between the smoothness of a function and the L^p norm of its Fourier transform.

Let us assume that g is a function defined in $\mathbb{R}^n$ and that its Fourier transform $\hat{g}$ belongs to L^1, hence the average contribution of high frequencies is less than $|\xi|^{-n}$ and it is well known that g is a uniformly continuous function. When $\hat{g} \in L^p$ for $1 < p \leq 2$, the average contribution of high frequencies can get larger and close to $|\xi|^{-n/p}$, consequently as the value of p increases g can develop stronger singularities, but by the Hausdorff-Young inequality g must stay in $L^{p'}$, where $\frac{1}{p'} + \frac{1}{p} = 1$. Finally, when $\hat{g} \in L^p$ for $p > 2$ then g is, in general, no longer a classical function but a distribution, and a very big mass of g can concentrate on a small "singular" set; for instance, if $\hat{g}$ is the constant function, then the whole mass of g concentrates at the origin in a Dirac's delta.

In the process of increasing p from 1 to ∞, or put in another way, as we let the contribution of high frequencies to increase, we may want to understand the process of explosion of g along a set. When does a function explode along a hyperplane in $\mathbb{R}^n$? For any $p > 1$ we can find a function g such that $\hat{g} \in L^p$ and that blows up along a hyperplane; to see this, take a Schwartz function $\varphi \in \mathbb{R}^{n-1}$ and define $\hat{g}(\xi', \xi_n) = \varphi(\xi')\langle\xi_n\rangle^{-1}$, where $\langle\xi_n\rangle = (1 + |\xi_n|^2)^{1/2}$. By Fourier inversion,

$$g(x', 0) = \int \varphi(\xi')\langle\xi_n\rangle^{-1} e(\langle\xi', x'\rangle)\, d\xi = \check{\varphi}(x') \int \langle\xi_n\rangle^{-1}\, d\xi_n,$$

but the last integral does not converge, hence a large mass of g can concentrate readily along a hyperplane as p increases. What does it happen if we replace the hyperplane by a sphere? If $g(x) = \varphi(x)|x - |x||^{-1/3}$, where φ is a continuous function supported in $B(0, 2)$, then $g \in L^2$ and by Plancherel $\hat{g} \in L^2$, hence there is a function whose Fourier transform belongs to L^2 but blows up along the sphere, however if $\hat{g} \in L^p$ for $1 < p < 2$ then the situation is much less clear. The curvature is here the important property that distinguishes the sphere from hyperplanes.

To understand what it happens over a sphere, we may try to get the bound $\|g\|_{L^1(S^{n-1})} \leq C\|\hat{g}\|_{L^p(\mathbb{R}^n)}$. By inversion, writing $f = \hat{g}$, and the properties of the Fourier transform, we can ask for the equivalent bound $\|\hat{f}\|_{L^1(S^{n-1})} \leq C\|f\|_{L^p(\mathbb{R}^n)}$. The operator $f \mapsto \mathcal{R}f = \hat{f}|_{S^{n-1}}$ is known as the restriction operator and it has been the object of intense research.

1.1 The Stein Restriction Conjecture

Around the seventies, Elias Stein posed the problem of determining the sharp range of exponents (p, q) for the bound $\|\hat{f}\|_{L^q(S^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ to hold. Instead of the restriction operator, one may prefer to work with the dual operator $\mathcal{E} = \mathcal{R}^*$, which acts on functions $g \in L^{q'}(S^{n-1})$. The dual operator $\mathcal{E}$ is given by the formal relationship

$$\int_{S^{n-1}} \hat{f} g \, d\sigma = \int_{\mathbb{R}^n} f (g \, d\sigma)^\wedge \, dx,$$

where $\mathcal{E}g = (g \, d\sigma)^\wedge$ and $d\sigma$ is the standard measure on the sphere, hence we can ask equivalently for the sharp range of exponents (q', p') for the bound $\|\mathcal{E}g\|_{L^{p'}} \lesssim \|g\|_{L^{q'}(S^{n-1})}$ to hold. The result of applying $\mathcal{E}$ to $g = 1$ is

$$(d\sigma)^\wedge(\xi) = \int_{S^{n-1}} e(-\langle \xi, x \rangle) \, d\sigma(x),$$

by the radial symmetry of $d\sigma$ it suffices to evaluate the integral when $\xi = te_n$, hence

$$(d\sigma)^\wedge(te_n) = \int_{S^{n-1}} e(-tx_n) \, d\sigma(x) = 2 \int_{|x|<1} \cos(2\pi t \sqrt{1 - |x|^2}) \, \frac{dx}{\sqrt{1 - |x|^2}};$$

expanding the phase function as $t\sqrt{1 - |x|^2} = t(1 - \frac{1}{2}|x|^2 + o(|x|^3))$, one sees that the function $\cos(2\pi t\sqrt{1 - |x|^2})$ remains roughly constant and equal to $\cos(2\pi t)$ whenever $|x| \ll \frac{1}{\sqrt{t}}$; otherwise, $\cos(2\pi t\sqrt{1 - |x|^2})$ oscillates quickly and the integral is negligible. Hence

$$(d\sigma)^\wedge(te_n) \approx 2 \int_{|x| \ll 1/\sqrt{t}} \cos(2\pi t \sqrt{1 - |x|^2}) \, \frac{dx}{\sqrt{1 - |x|^2}} \approx \frac{\cos(2\pi t)}{t^{\frac{n-1}{2}}}, \qquad \text{if } t \gg 1. \tag{1-1}$$

This is a rather crude estimation, but gives us quickly the right rate of decay $|(d\sigma)^\wedge(\xi)| = O(|\xi|^{-\frac{n-1}{2}})$. This method is a rudimentary form of the principle of stationary phase, and more precise asymptotic expansions are possible; see [62, ch. 8]. Since $g = 1$ belongs to every $L^{q'}(S^{n-1})$ and $|\mathcal{E}g(\xi)| \approx |\xi|^{-\frac{n-1}{2}}$ for $|\xi| \gg 1$, the extension operator $\mathcal{E}$ can only be bounded for $p' > \frac{2n}{n-1}$.

In what follows we will make extensive use of the uncertainty principle, that in its most rudimentary form says that a localized function must have a delocalized Fourier transform.

Therefore, it seems convenient now to give a more detailed explanation. Suppose that a function f is supported in a ball of radius $r < 1$, and we want to calculate its Fourier transform. If we test f against sufficiently low frequencies ξ, *i.e.* if we evaluate $\int f(x)e(-\langle x,\xi\rangle)\,dx$, so that the wave length $\frac{1}{|\xi|}$ is $\geq r$ and $e(-\langle x,\xi\rangle)\approx 1$ in B_r, then $\hat{f}(\xi)$ is roughly constant and equal to $\int f$ in a large ball of radius r^{-1}. The reader may use this to convince himself that a function localized in the ellipse $\sum \frac{x_i^2}{\lambda_i^2} \leq 1$ has a Fourier transform localized in the dual ellipse $\sum(\lambda_i x_i)^2 \leq 1$. There are several ways of formalizing and strengthening this principle. In the following chapter, we will give an example of a rigorous proof using the uncertainty principle.

We test now the action of $\mathcal{E}$ over the characteristic function of a cap or ball $\theta \subset S^{n-1}$ of radius $\delta \ll 1$. Since the sphere looks locally as a paraboloid, then θ is contained in a rectangle, not necessarily aligned with the coordinate axes, of dimensions $\delta\times\cdots\times\delta\times\delta^2$. By the uncertainty principle, $\mathcal{E}g = (\mathbb{1}_\theta d\sigma)^\wedge$ is essentially supported in a dual rectangle of dimensions $\delta^{-1}\times\cdots\times\delta^{-1}\times\delta^{-2}$ and in this rectangle the value of $|\mathcal{E}g|$ is approximately δ^{n-1}, the volume of the cap. To prove this, let us assume that the center of θ is the north pole, then

$$\begin{aligned}|(\mathbb{1}_\theta d\sigma)^\wedge(\xi)| &= \left| e(-\xi_n)\int_{|x|<\delta} e(-\langle \xi', x\rangle - \xi_n\sqrt{1-|x|^2}+\xi_n)\,\frac{dx}{\sqrt{1-|x|^2}}\right|,\\ &> \left|\int_{|x|<\delta} \cos(-\langle \xi', x\rangle - \xi_n\sqrt{1-|x|^2}+\xi_n)\,\frac{dx}{\sqrt{1-|x|^2}}\right|,\end{aligned}$$

if ξ lies inside the set $|\xi'|\leq c\delta^{-1}$ and $|\xi_n|\leq c\delta^{-2}$ for $c \ll 1$, then $|\langle\xi',x\rangle+\xi_n(\sqrt{1-|x|^2}-1)|\leq \frac{1}{10}$. Hence, $|(\mathbb{1}_\theta d\sigma)^\wedge(\xi)| \gtrsim \delta^{n-1}$ in a rectangle of dimensions $\delta^{-1}\times\cdots\times\delta^{-1}\times\delta^{-2}$. Assuming as true the inequality $\|\mathcal{E}g\|_{L^{p'}} \lesssim \|g\|_{L^{q'}(S^{n-1})}$ for every $g\in L^{q'}(S^{n-1})$, we get $\delta^{n-1-\frac{n+1}{p'}} \lesssim \delta^{\frac{n-1}{q'}}$ for $\delta \ll 1$, that forces $p' \geq q\frac{n+1}{n-1}$. This example is known as the Knapp example, and together with the example $g=1$, we are led to the Stein restriction conjecture:

The restriction operator is bounded as long as $p' > \frac{2n}{n-1}$ and $p' \geq q\frac{n+1}{n-1}$.

The conjecture holds in $\mathbb{R}^2$, thanks to the work of Fefferman [30] and Zygmund [77], but remains open in higher dimensions. Analogous results and conjectures for more general oscillatory integrals were given by Hörmander [39].

It is not coincidence that this conjecture arouse with a renewed interest on the problem of Bochner-Riesz summability and the study of dispersive equations; the reader may consult the spirit at that moment in [32]. Regarding Bochner-Riesz summability, a basic question in harmonic analysis is the sense in which we understand that $f(x) = \int \hat{f}(\xi)e(\langle x,\xi\rangle)\,d\xi$. The Bochner-Riesz mean is a method of summation of the Fourier transform, that allows, under certain conditions, to recover the original function and consists in taking the limit

$$\int (1-|\xi/R|^2)_+^\alpha \hat{f}(\xi)e(\langle x,\xi\rangle)\,d\xi \xrightarrow{R\to\infty} f(x),$$

where $t_+ = \max\{0,t\}$ and $\alpha > 0$. Natural questions are: how do we understand this limit? does the limit make sense in L^p? This latter question is equivalent to the L^p boundedness of the operator $S^\alpha f = ((1-|\xi|^2)_+^\alpha \hat{f})^\vee$, that is, to the estimate $\|S^\alpha f\|_p \lesssim \|f\|_p$. Fefferman showed in [30, 33] that $\alpha > \max\{n|\frac{1}{p}-\frac{1}{2}|-\frac{1}{2}, 0\}$ is a necessary condition, and it is conjectured that it is also sufficient. This was proven in $n=2$ by Carleson and Sjölin [18].

The effect of the spherical boundary of $(1-|\xi|^2)_+^\alpha$ is reflected in the asymptotic expansion

$$K^\alpha(x) := ((1-|\xi|^2)_+^\alpha)^\vee \approx \frac{\cos(2\pi|x|)}{|x|^{\frac{n+1}{2}+\alpha}} = \frac{e(|x|)}{2|x|^{\frac{n+1}{2}+\alpha}} + \frac{e(-|x|)}{2|x|^{\frac{n+1}{2}+\alpha}}. \tag{1-2}$$

The operator $S^\alpha f$ equals the convolution $K^\alpha * f$, and it is convenient to decompose the kernel K^α dyadically into pieces $K_j^\alpha(x) = \varphi_j K^\alpha(x)$, where φ_j is supported in $|x| \sim 2^j$ for $j \geq 0$. Furthermore, being $K_j^\alpha * f$ a local operator that commutes with translations, we can assume that f is supported in a ball of radius 2^j centered at the origin. Hence, whenever $|x| \sim 2^j$ we use the Taylor expansion around $y=0$ to get $|x-y| = |x| - \langle \frac{x}{|x|}, y\rangle + O(|y|^2)$; hence,

$$K_j^\alpha * f(x) = \int K_j^\alpha(x-y) f(y)\, dy \approx \frac{e(\pm|x|)}{|x|^{\frac{n+1}{2}+\alpha}} \hat{f}(\pm\frac{x}{|x|}).$$

The last formula, connecting the Bochner-Riesz means and the restriction operator, was exploited by Fefferman [30, 31] to get partial progress on the problem of summability from estimates for the restriction operator, and conversely by Tao [64] to prove that the Bochner-Riesz conjecture implies the restriction conjecture.

As regards dispersive equations, to be specific, let us focus on the Schrödinger equation $iu_t + \Delta u = 0$. The space-time Fourier transform is $-2\pi\tau\hat{u} - 4\pi^2|\xi|^2\hat{u} = 0$, hence $\hat{u}$ must be a measure supported in the paraboloid $\tau = -2\pi|\xi|^2$, and u is the inverse Fourier transform of this measure. To avoid technical annoyances, let us consider instead a truncated paraboloid. It is natural then to study the extension operator associated not to the sphere, but to the paraboloid. It turns out that the necessary conditions for the $L^{p'} \to L^{q'}$ boundedness of the extension operator over a truncated paraboloid are the same as for the sphere. Moreover, we can replace the sphere by any other compact elliptic surface, and the range of conjectured exponents is the same. A partition of unity allows us to focus on a neighborhood of a point in the hypersurface that, after a change of coordinates if necessary, can be written as the graph of a function φ, so the extension operator can be written as

$$\mathcal{E}f(x', x_n) = \int_{U\subset\mathbb{R}^{n-1}} f(\xi) e(-\langle x', \xi\rangle - x_n\varphi(\xi))\, d\xi. \tag{1-3}$$

Here, we have absorbed the harmless term $\sqrt{1+|\nabla\varphi|^2}$ in f. The restriction conjecture can be naturally extended to other model phases φ. Since we can identify the hypersurface S with $U \subset \mathbb{R}^{n-1}$, one usually works over U; the lift of a set $V \subset U$ to S is denoted by $\tilde{V}$, and likewise the lift of a point $a \in U$ is denoted by $\tilde{a}$. We can carry naturally a function f defined in U to a measure supported in $\tilde{U}$, and we still denote this lifting as $f d\sigma$.

The first big step towards the restriction conjecture in all dimensions was given by Tomas [70] and Stein (unpublished), who proved it in the optimal range of exponents whenever $p' \geq 2\frac{n+1}{n-1}$. The proof relies on L^2 methods. We must show $\|\hat{f}\|_{L^2(S^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$, so expanding the square and using properties of the Fourier transform

$$\|\hat{f}\|_{L^2(S)}^2 = \int \bar{\hat{f}}\hat{f}\, d\sigma = \int \bar{f}(-x)(f * \widehat{d\sigma})(x)\, dx \leq \|f\|_{L^p}\|f * \widehat{d\sigma}\|_{L^{p'}},$$

hence it suffices to prove $\|f * \widehat{d\sigma}\|_{L^{p'}} \lesssim \|f\|_{L^p}$ for $p' \geq 2\frac{n+1}{n-1}$ (this is the TT^* method). To prove this, we use interpolation of operators. On the one hand, the operator $f * \widehat{d\sigma}$ is well behaved if $p' = \infty$ and we can worsen it to $f * (\langle\xi\rangle^{\frac{n-1}{2}}\widehat{d\sigma})$ without ruining the $L^1 \to L^\infty$ boundedness; this follows from (1-1) and Young's inequality. On the other hand, $f * \widehat{d\sigma}$ is not well behaved if $p' = 2$, but we can enforce $L^2 \to L^2$ boundedness improving the operator to $f * (\langle\xi\rangle^{-1}\widehat{d\sigma})$. To see this, notice that $\langle\xi\rangle^{-1}\widehat{d\sigma} \approx \widehat{\mathbb{1}}_{B_1}$ (compare (1-1) with (1-2) for $\alpha = 0$), therefore by Plancherel $\|f * (\langle\xi\rangle^{-1}\widehat{d\sigma})\|_2 \approx \|\hat{f}\mathbb{1}_{B_1}\|_2 \leq \|f\|_2$. Hence, we get the bound $\|f * (\langle\xi\rangle^{\alpha}\widehat{d\sigma})\|_{p'} \lesssim \|f\|_p$ for $\alpha = \frac{n-1}{2} - \frac{n+1}{p'}$ and $2 \leq p' \leq \infty$, so we simply consider the case $\alpha = 0$, which implies $p' = 2\frac{n+1}{n-1}$. See the whole argument in [62, chp. 9].

The Tomas-Stein estimate was extended by Strichartz [63] to quadric hypersurfaces, not necessarily the paraboloid. These estimates have had a deep impact on the theory of well-posedness of dispersive equations.

The field was relatively dormant until the work of Bourgain [3], who pushed the restriction conjecture beyond L^2 methods. There are two important observations in his work. The first is a wave-packet decomposition used by Fefferman and Córdoba, see e.g. [31, 20], and the second is an induction on scales argument. We present here a sketch of the argument to get $L^{p'} \to L^{p'}$ bounds, however in the next chapter a similar, and rigorous argument, will be given.

We recall our definition of $\mathcal{E}f$ in (1-3), where the function f is now defined in a set $U \subset \mathbb{R}^{n-1}$. For simplicity, we assume that $S = \tilde{U}$ is a neighborhood of the north pole in the sphere. We begin by dividing U into cubes α, or caps as they are usually named, of side-length δ and center $c_\alpha \in U$, which likewise induce a partition of S; therefore, we can write $\mathcal{E}f = \sum_\alpha \mathcal{E}f_\alpha$, where $f_\alpha = f\mathbb{1}_\alpha$. The function $|\mathcal{E}f_\alpha|^2 = [f_\alpha d\sigma * (\overline{f_\alpha d\sigma}(-\cdot))]^\wedge$ is the Fourier transform of a function supported in a rectangle R_α centered at the origin of dimensions $\delta \times \cdots \times \delta \times \delta^2$. Let us denote by T_α the dual rectangle to R_α and we depart slightly from our conventions to write $|T_\alpha| := \int \zeta_{T_\alpha}$ for some bump function ζ_{T_α}. If we choose ζ_{T_α} such that $\check{\zeta}_{T_\alpha} = |T_\alpha|$ in R_α, then we can write

$$f_\alpha d\sigma * (\overline{f_\alpha d\sigma}(-\cdot)) = \frac{1}{|T_\alpha|} f_\alpha d\sigma * (\overline{f_\alpha d\sigma}(-\cdot))\check{\zeta}_{T_\alpha}; \tag{1-4}$$

by taking Fourier transform we get the *reproducing formula* $|\mathcal{E}f_\alpha|^2 = \frac{1}{|T_\alpha|}|\mathcal{E}f_\alpha|^2 * \zeta_{T_\alpha}$. The function ζ_{T_α} is roughly supported in the tube T_α centered at the origin of dimensions $\delta^{-1} \times \cdots \times \delta^{-1} \times \delta^{-2}$ pointing in the direction of the normal vector to S at $\tilde{c}_\alpha$. Hence, $|\mathcal{E}f_\alpha|$ is approximately constant on translations of the tube T_α, or from other point of view, each $|\mathcal{E} f_\alpha|^2$ can be seen as the weighted sum $\sum_\nu |\mathcal{E} f_\alpha|^2(\nu)\zeta_{T_\alpha+\nu}$, where the sets $T_\alpha+\nu$ are a tiling of $\mathbb{R}^n$; this kind of decomposition is known as *wave-packet* decomposition.

If x is constrained to be in a ball of radius δ^{-1} centered at x_0, one can write $\mathcal{E} f_\alpha$ as

$$\begin{aligned}\mathcal{E} f_\alpha(x) &= \int_\alpha f(\xi)e(-\langle (x-x_0)+x_0',\xi\rangle - ((x_n-x_{0,n})+x_{0,n})\varphi(\xi))\,d\xi,\\ &\approx \mathcal{E} f_\alpha(x_0)e(-\langle x-x_0,\tilde{c}_\alpha\rangle).\end{aligned} \tag{1-5}$$

Hence,

$$\int_{B_{\delta^{-1}}} |\mathcal{E} f|^{p'}\,dx = \int_{B_{\delta^{-1}}} |\sum_\alpha \mathcal{E} f_\alpha|^{p'}\,dx \approx \int_{B_{\delta^{-1}}} |\sum_\alpha \mathcal{E} f_\alpha(x_0)e(-\langle x-x_0,\tilde{c}_\alpha\rangle)|^{p'}\,dx. \tag{1-6}$$

The term $\sum_\alpha \mathcal{E} f_\alpha(x_0)e(-\langle x-x_0,\tilde{c}_\alpha\rangle)$ resembles the extension operator applied to the function

$$g = \delta^{-(n-1)}\sum_\alpha \mathcal{E} f_\alpha(x_0)e(\langle x_0,\tilde{c}_\alpha\rangle)\mathbb{1}_\alpha.$$

We need the following lemma, which will also be used in the next chapter.

Lemma 1.1. *If the extension operator is defined over the hypersurface* $S=\{(\xi,\varphi(\xi))\mid \xi\in B_1\}$*, then*

$$\|\mathcal{E}g\|_{L^2(\mathbb{R}^{n-1}\times[-R,R])} \le CR^{1/2}\|g\|_2 \tag{1-7}$$

where C *does not depend on* g *or* R*.*

Proof. We prove that $\|\mathcal{E}g\|_{L^2(B_\rho\times[-R,R])}\le CR^{1/2}\|g\|_2$, with C independent of $\rho\ge 1$. Using a bump function ζ_ρ adapted to $B_\rho\times[-R,R]$ and Plancherel we obtain

$$\|\mathcal{E}g\|_{L^2(B_\rho\times[-R,R])} \lesssim \|\mathcal{E}g\zeta_\rho\|_2 = \|gd\sigma * \check{\zeta}_\rho\|_2;$$

where we choose ζ_ρ such that its Fourier transform is compactly supported in the set $T_{\rho,R} := B_{\rho^{-1}}\times[-R^{-1},R^{-1}]$. We prove the lemma by interpolation between $L^1\to L^1$ and $L^\infty\to L^\infty$. We have thus

$$\|gd\sigma * \check{\zeta}_\rho\|_1 \le \int |g(\xi)| \int |\check{\zeta}_\rho(\eta'-\xi,\eta_n-\varphi(\xi))|\,d\eta d\xi \lesssim \|g\|_1.$$

For the other point of interpolation we have

$$\begin{aligned}\|gd\sigma * \check{\zeta}_\rho\|_\infty &\le \sup_\eta \int |g(\xi)\check{\zeta}_\rho(\eta'-\xi,\eta_n-\varphi(\xi))|\,d\xi\\ &\lesssim R\rho^{n-1}\|g\|_\infty \sup_\eta \int \mathbb{1}_{T_{\rho,R}}(\eta'-\xi,\eta_n-\varphi(\xi))\,d\xi\\ &\lesssim R\|g\|_\infty,\end{aligned}$$

which concludes the proof. □

Interpolating (1-7), for $R = \delta^{-1}$, and the Tomas-Stein inequality, *i.e.* $\|\mathcal{E}g\|_{L^{2\frac{n+1}{n-1}}(B_{\delta^{-1}})} \leq C\|g\|_2$, we get

$$\|\mathcal{E}g\|_{L^{p'}(B_{\delta^{-1}})} \lesssim \delta^{\frac{n-1}{4}-\frac{n+1}{2p'}}\|g\|_2.$$

We replace this bound in (1-6) to get

$$\Big(\int_{B_{\delta^{-1}}} |\mathcal{E}f|^{p'}\,dx\Big)^{1/p'} \leq \delta^{\frac{n-1}{4}-\frac{n+1}{2p'}}\|g\|_2 = \delta^{-\frac{n-1}{4}-\frac{n+1}{2p'}}\Big(\sum_\alpha |\mathcal{E}f_\alpha(x_0)|^2\Big)^{1/2}. \tag{1-8}$$

Summing over all the balls $B_{\delta^{-1}}$ covering $B_{\delta^{-2}}$ we have

$$\|\mathcal{E}f\|_{L^{p'}(B_{\delta^{-2}})} \lesssim \delta^{-\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p'})}\|\big(\sum_\alpha |\mathcal{E}f_\alpha|^2\big)^{1/2}\|_{L^{p'}(B_{\delta^{-2}})}, \qquad \text{for } 2 \leq p' \leq 2\frac{n+1}{n-1}. \tag{1-9}$$

The term on the right side of the inequality is known as a *square function*, and it is conjectured that

$$\|\mathcal{E}f\|_{L^{p'}(B_{\delta^{-2}})} \lesssim \|\big(\sum_\alpha |\mathcal{E}f_\alpha|^2\big)^{1/2}\|_{L^{p'}(B_{\delta^{-2}})}, \qquad \text{for } 2 \leq p' < \frac{2n}{n-1}. \tag{1-10}$$

To estimate the square function we use duality, which is allowed because $r = p'/2 > 1$. To simplify, let us replace ζ_{T_α} by $\mathbb{1}_{T_\alpha}$. Using $|\mathcal{E}f_\alpha|^2 = \frac{1}{|T_\alpha|}|\mathcal{E}f_\alpha|^2 * \zeta_{T_\alpha}$, Fubini and the identity $\mathbb{1}_{T_\alpha}(x) = \mathbb{1}_{T_\alpha}(-x)$ we get

$$\begin{aligned}\|\sum_\alpha |\mathcal{E}f_\alpha|^2\|_{L^{p'/2}(B_{\delta^{-2}})} &= \sup_{\substack{\|w\|_{r'}=1\\ \operatorname{supp} w\subset B_{\delta^{-2}}}} \Big|\int \big(\sum_\alpha |\mathcal{E}f_\alpha|^2\big) w\,dx\Big| \\ &\approx \sup_{\substack{\|w\|_{r'}=1\\ \operatorname{supp} w\subset B_{\delta^{-2}}}} \Big|\sum_\alpha \int \frac{1}{|T_\alpha|}|\mathcal{E}f_\alpha|^2 \mathbb{1}_{T_\alpha} * w\,dx\Big|.\end{aligned}$$

We estimate the last integral as

$$\Big|\sum_\alpha \int \frac{1}{|T_\alpha|}|\mathcal{E}f_\alpha|^2 \mathbb{1}_{T_\alpha} * w\,dx\Big| \leq \delta^{n+1}\sum_\alpha \sup|\mathbb{1}_{T_\alpha} * w| \int_{B_{2\delta^{-2}}} |\mathcal{E}f_\alpha|^2\,dx.$$

By (1-7) we have $\|\mathcal{E}f_\alpha\|_{L^2(B_{2\delta^{-2}})} \lesssim \delta^{-1}\|f_\alpha\|_2$. Plugging in the above equation and using Hölder we get

$$\begin{aligned}\Big|\sum_\alpha \int_{B_{\delta^{-2}}} \frac{1}{|T_\alpha|}|\mathcal{E}f_\alpha|^2 \mathbb{1}_{T_\alpha} * w\,dx\Big| &\lesssim \delta^{n-1}\sum_\alpha \sup|\mathbb{1}_{T_\alpha} * w|\|f_\alpha\|_2^2 \\ &\leq \delta^{n-1}\Big(\sum_\alpha \|\mathbb{1}_{T_\alpha} * w\|_\infty^{r'}\Big)^{1/r'}\Big(\sum_\alpha \|f_\alpha\|_2^{p'}\Big)^{2/p'}.\end{aligned} \tag{1-11}$$

Since T_α remains roughly unaltered after tilting it by angles less than δ, one may write

$$\Big(\sum_\alpha \|\mathbb{1}_{T_\alpha} * w\|_\infty^{r'}\Big)^{1/r'} \approx \delta^{-\frac{n-1}{r'}}\Big(\int_{\tilde{U}\subset S^{n-1}} \|\mathbb{1}_{T_\alpha} * w\|_\infty^{r'}\Big)^{1/r'},$$

where in the last integral we are summing over all the caps α of radius δ and center $\tilde{c}_\alpha \in \tilde{U}$.

The term $\sup_x |\mathbb{1}_{T_\alpha} * w(x)|$ is known as the Kakeya maximal function, however it is usually rescaled and written for tubes $T_\omega^\delta \subset \mathbb{R}^n$ of unit length, radius $\delta < 1$ and with direction $\omega \in S^{n-1}$, so that the classical Kakeya maximal function is $\mathcal{K}_\delta w(\omega) := \delta^{-(n-1)} \sup_x |\mathbb{1}_{T_\omega^\delta} * w|$. The Kakeya maximal function conjecture states that

$$\Big(\int_{S^{n-1}} \mathcal{K}_\delta w(\omega)^{r'}\, d\sigma(\omega)\Big)^{1/r'} \lesssim \delta^{1-\frac{n}{r'}} \|w\|_{r'}, \qquad \text{for } r' < n, \text{ all } \delta < 1 \text{ and } \operatorname{supp} w \subset B_1. \quad (1\text{-}12)$$

To see that the conjecture is sharp, test $\mathcal{K}_\delta$ against $f = \mathbb{1}_{B_\delta}$ and $f = \mathbb{1}_{E^\delta}$, where E^δ is the δ-neighborhood of a Besicovitch set, *i.e.*, a set of Lebesgue measure zero containing a line of unit length in every direction. The conjecture is still wide open, and a wealth of research has been directed to prove it, or related conjectures, see *e.g.* [57, 21, 47, 41, 9, 43]. The Kakeya maximal function conjecture was settled by Córdoba in [20] for the case of the plane. In higher dimensions, the conjecture was verified by Wolff [72] in the range $r' \leq \frac{n+2}{2}$. Later, Katz and Tao [42] got the range $r' \leq \frac{4n+3}{7}$, which improves Wolff's result for $n \geq 9$.

Let us assume the Kakeya maximal function conjecture as true, so after rescaling we get

$$\Big(\sum_\alpha \|\mathbb{1}_{T_\alpha} * w\|_\infty^{r'}\Big)^{1/r'} \lesssim \delta^{-\frac{2}{p'}-(n-1)};$$

recall that $\|w\|_{r'} = 1$. We replace this bound in (1-11) and use Hölder inequality to reach

$$\begin{aligned} |\sum_\alpha \int_{B_{\delta^{-2}}} \frac{1}{|T_\alpha|} |\mathcal{E} f_\alpha|^2 \mathbb{1}_{T_\alpha} * w\, dx| &\lesssim \delta^{-\frac{2}{p'}} \Big(\sum_\alpha \|f_\alpha\|_2^{p'}\Big)^{2/p'} \\ &\leq \delta^{-\frac{2}{p'}+(n-1)(1-\frac{2}{p'})} \Big(\sum_\alpha \|f_\alpha\|_{p'}^{p'}\Big)^{2/p'} \\ &= \delta^{-\frac{2n}{p'}+n-1} \|f\|_{p'}^2. \end{aligned}$$

Returning to (1-9), we replace our square function estimate to get

$$\|\mathcal{E} f\|_{L^{p'}(B_{\delta^{-2}})} \lesssim \delta^{-\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p'})+\frac{n-1}{2}-\frac{n}{p'}} \|f\|_{p'} = \delta^{\frac{n-1}{4}-\frac{n+1}{2p'}} \|f\|_{p'}.$$

Since Bourgain did not have at hand the optimal ranges of the square function and Kakeya maximal function conjectures, he only obtained a bound $\|\mathcal{E} f\|_{L^{p'}(B_{\delta^{-2}})} \lesssim_{\alpha_p} \delta^{-\alpha_p} \|f\|_{p'}$ for some $\alpha_p > 0$ such that $\alpha_p = 0$ for $p' = 2\frac{n+1}{n-1}$. This is a local result, because we are integrating over a ball $B_{\delta^{-2}}$, however if α_p is small enough for some p close to the Tomas-Stein exponent, Bourgain invented a technique, now called ϵ-removal lemma, to trade the loss $\delta^{-\alpha_p}$ of the local result by a global result in a narrower range $p' \geq 2\frac{n+1}{n-1} - \epsilon_n$, for some $\epsilon_n > 0$, improving so over the Tomas-Stein exponent.

If the square function and Kakeya conjectures were true, then we would get the local bound $\|\mathcal{E} f\|_{L^{\frac{2n}{n-1}}(B_{\delta^{-2}})} \lesssim_\epsilon \delta^{-\epsilon} \|f\|_{\frac{2n}{n-1}}$, and applying a refinement of the ϵ-removal lemma due to Tao, see [64], we would get the global bound without ϵ losses for $p' > \frac{2n}{n-1}$.

1.2 The Multilinear Theory

A first hint towards the multilinear theory is contained in the proof of the restriction conjecture in $\mathbb{R}^2$. We want to prove the bound $\|(fd\sigma)^\wedge\|_{p'} \leq C\|f\|_\infty$ for $p' > 4$, then we use Plancherel and Hausdorff-Young to get

$$\begin{aligned}\|(fd\sigma)^\wedge\|_{p'} = \|(fd\sigma)^\wedge(fd\sigma)^\wedge\|_{p'/2}^{1/2} = \|(fd\sigma * fd\sigma)^\wedge\|_{p'/2}^{1/2} \\ \leq \|fd\sigma * fd\sigma\|_r^{1/2} \leq \|f\|_\infty \|d\sigma * d\sigma\|_r^{1/2},\end{aligned}$$

where $r = (p'/2)' < 2$. If the curve C has non-zero curvature everywhere, then it is possible to show that $\|d\sigma * d\sigma\|_r < \infty$ for $r < 2$, proving so the bound $L^\infty \to L^{p'}$ in the conjectured range $p' > 4$. Let us prove the inequality $\|d\sigma * d\sigma\|_r < \infty$ for a truncated parabola C, whose corresponding measure can be written as $d\sigma = \mathbb{1}_{\{|x_1|\leq 1\}}\delta(x_2 - x_1^2)$ for $x \in \mathbb{R}^2$. Hence,

$$\begin{aligned}d\sigma * d\sigma(x) &= \int \delta(x_2 - y_2 - (x_1 - y_1)^2)\delta(y_2 - y_1^2)\mathbb{1}_{\{|x_1 - y_1|\leq 1\}}\mathbb{1}_{\{|y_1|\leq 1\}}\, dy \\ &\leq \mathbb{1}_{B_3}(x)\int \delta(x_2 - y_2 - (x_1 - y_1)^2)\delta(y_2 - y_1^2)\, dy.\end{aligned}$$

We make the $2-1$ change of variables $z_1 = y_2 - y_1^2$ and $z_2 = x_2 - y_2 - (x_1 - y_1)^2$ to get

$$d\sigma * d\sigma(x) \leq 2\mathbb{1}_{B_3}(x)\int \delta(z_1)\delta(z_2)\frac{dz}{2|2y_1 - x_1|} = \mathbb{1}_{B_3}(x)\frac{2}{\sqrt{2x_2 - x_1^2}} \lesssim \mathbb{1}_{B_3}(x)\frac{1}{\sqrt{d(x, 2C)}},$$

where $d(x, 2C)$ is the distance from x to the parabola $2C = \{x_2 = \frac{1}{2}x_1^2\}$. Thus, the function $d\sigma * d\sigma$ belongs to L^r for $1 \leq r < 2$.

After further progress on the restriction conjecture, *e.g.* [4, 7, 8, 55, 56], enriched by the study of non-linear dispersive equations, *e.g.* [46, 45], the time was ripe for the development of the bilinear theory by Tao, Vargas and Vega [68]. They considered the inequality

$$\|\mathcal{E}f\mathcal{E}g\|_{p'/2} \lesssim \|f\|_{L^{q'}(S_1)}\|g\|_{L^{q'}(S_2)},$$

where S_1 and S_2 are subsets of the paraboloid, separated by a distance ~ 1. The bilinear inequality follows by Hölder inequality from the linear one, and conversely the linear extension estimate for (q', p') would follow if we were able to choose $S_1 = S_2$ and $f = g$, but the separation condition prevent us from that, so we may expect that the bilinear estimate holds for the same range as for the linear one; however, they showed that the bilinear estimate is indeed more stable due to transversal intersections of tubes, enhancing the range of necessary conditions to

$$\begin{aligned}p' &\geq \frac{2n}{n-1}, \\ \frac{n+2}{p'} + \frac{n}{q'} &\leq n, \\ \frac{n+2}{p'} + \frac{n-2}{q'} &\leq n-1.\end{aligned}$$

We notice that the last two conditions coincide if $q' = 2$, and in this case $p' \geq 2\frac{n+2}{n}$, which extends the Tomas-Stein range $p' \geq 2\frac{n+1}{n-1}$. The bilinear inequality for $q' = 2$ was known as the Klainerman-Machedon conjecture. One of the main results of Tao, Vargas and Vega was roughly that the linear estimate $L^{q'} \to L^{p'}$ follows, in the conjectured linear range, from the bilinear estimate $L^{q'} \times L^{q'} \to L^{p'/2}$.

Wolff brought to the field fresh ideas from combinatorics and proved the Klainerman-Machedon conjecture, up to logarithmic losses, for conical surfaces [75]. He used the wave-packet decomposition and modified the induction on scales by introducing a new scale $\delta^{-2+\epsilon}$ between δ^{-1} and δ^{-2}. Then, he split the tubes passing through a ball of radius $B_{\delta^{-2+\epsilon}}$ into tubes with big and small contribution. The tubes with big contribution are small in number, so the inductive hypothesis provides an allowable contribution. Since the norm we must control is $L^{\frac{n+2}{n}}$, the tubes with small contribution can be controlled by interpolation between the norms L^1 and L^2. The norm L^1 is easily controlled by Hölder, but the L^2 side requires heavy combinatorics.

Tao went on proving the case of the paraboloid [66], establishing the restriction conjecture for elliptic surfaces in the range $p' > 2\frac{n+2}{n}$. This is the best we can expect from bilinear methods.

If we consider multilinear estimates, instead of bilinear estimates, then the stability of the operator increases even more. We know that the linear theory is meaningless in the case of hyperplanes in $\mathbb{R}^n$, as we discussed at the very beginning; however, if we apply the extension operator to n measures $\mu_i = f_i\, dx_1 \cdots \widehat{dx_i} \cdots dx_n$ ($\widehat{\cdot}$ means that we omit this term) carried by the hyperplanes $S_i = \{x \mid x_i = 0\}$, then the corresponding multilinear operator would be

$$\int |\widehat{\mu}_1(\pi_1(\xi)) \cdots \widehat{\mu}_n(\pi_n(\xi))|^{p'/n}\, d\xi \leq C \|f_1\|_{q'} \cdots \|f_n\|_{q'}, \tag{1-13}$$

where π_i is the projection to the hyperplane S_i (we used the fact that $\widehat{\mu}_i(\pi_i(\xi)) = \widehat{\mu}_i(\xi)$). But this type of inequalities has been known for a long time, namely, this is the Loomis-Whitney inequality [49].

Theorem 1.2. *If the functions $h_i : \mathbb{R}^{n-1} \to \mathbb{R}_+$, for $i = 1, \ldots, n$, are integrable, then*

$$\int_{\mathbb{R}^n} (h_1(\pi_1(\xi)) \cdots h_n(\pi_n(\xi)))^{1/(n-1)}\, d\xi \leq C(\|h_1\|_1 \cdots \|h_n\|_1)^{1/(n-1)}. \tag{1-14}$$

The inequality can be proven by induction, starting from the case $n = 2$. Now, replacing h_i by $|\widehat{\mu}_i|^2$ and using Plancherel, the Loomis-Whitney inequality allows us to establish the inequality (1-13) for $p' = \frac{2n}{n-1}$ and $q' = 2$; however, observe that the linear estimate $L^2 \to L^{\frac{2n}{n-1}}$ for the extension operator associated to hyperplanes is false. Amazingly, these n-multilinear estimates do not require any curvature assumptions.

The multilinear theory was developed by Bennett, Carbery and Tao [2] and they found out that the fundamental assumption behind multilinearity was not curvature, but transversality. They proved the corresponding multilinear conjecture, up to logarithmic losses.

Theorem 1.3. *If we are given n smooth compact hypersurfaces with boundary S_k, such that for every n points $\xi_k \in S_k$ the corresponding normal vectors $N(\xi_k)$ to S_k satisfy the transversality condition $|\det(N(\xi_1)\cdots N(\xi_n))| \geq \theta > 0$, then for $p' \geq \frac{2n}{n-1}$ and $p' \geq q\frac{n}{n-1}$ it holds*

$$\int_{B_R} \prod_{k=1}^{n} |\mathcal{E} f_k|^{p'/n}\, d\xi \leq C_\epsilon R^\epsilon \prod_{k=1}^{n} \|f_k\|_{q'}^{p'/n}, \tag{1-15}$$

for every $R \geq 1$ and $\epsilon > 0$.

It suffices to prove the end-point $q' = 2$ and $p' = \frac{2n}{n-1}$, and the proof is the multilinear adaptation of the induction on scales presented above, but now the linear Kakeya inequality (1-12) is replaced by a multilinear Kakeya estimate.

Theorem 1.4. *If we are given n collections of tubes $\{T_{k,i}\}_{i=1,\ldots,N_k}$, for $k = 1,\ldots,n$ and with diameter 1, whose direction vectors $v_{k,i}$ satisfy $|\det(v_{1,i_1}\cdots v_{n,i_n})| \geq \theta > 0$ for every $i_1,\ldots i_n$, then*

$$\int_{B_R} (\prod_{k=1}^{n} \sum_i \mathbb{1}_{T_{k,i}})^{1/(n-1)}\, dx \leq C_\epsilon R^\epsilon \prod_{k=1}^{n} N_k^{1/(n-1)}, \tag{1-16}$$

where N_k is the number of tubes in $\{T_{k,i}\}_{i=1,\ldots,N_k}$.

The theorem does not exclude the possibility of repeating tubes. After the spectacular resolution of the Kakeya conjecture for finite fields by Dvir [22], Guth [35] took wholly new ideas from algebraic geometry, together with the polynomial method in the paper of Dvir, and removed the term R^ϵ from the multilinear Kakeya inequality and gave the sharp dependence on transversality.

Theorem 1.5. *If we are given n collections of tubes $\{T_{k,i}\}_{i=1,\ldots,N_k}$, for $k = 1,\ldots,n$ and with diameter 1, whose direction vectors $v_{k,i}$ satisfy $|\det(v_{1,i_1}\cdots v_{n,i_n})| \geq \theta > 0$ for every $i_1,\ldots i_n$, then*

$$\int (\prod_{k=1}^{n} \sum_i \mathbb{1}_{T_{k,i}})^{1/(n-1)}\, dx \leq C\theta^{-\frac{1}{n-1}} \prod_{k=1}^{n} N_k^{1/(n-1)}, \tag{1-17}$$

where N_k is the number of tubes in $\{T_{k,i}\}_{i=1,\ldots,N_k}$.

Unfortunately, the sharp constant $C\theta^{-\frac{1}{n-1}}$ does not carry over immediately to the multilinear extension inequality, due to the many inefficiencies in the inductive process. See also [36, 17] for alternative proofs of the multilinear Kakeya estimate.

Unlike the bilinear theory, some time had to pass for the multilinear estimates to be useful in the linear restriction problem. Bourgain and Guth [16] achieved this by exploiting the dichotomy between transversal and non-transversal contributions. We will sketch the Bourgain-Guth argument to get $L^{p'} \to L^{p'}$ bounds of the restriction operator in the range $p' > \frac{2n}{n-1}$.

Recall the remarks after our definition of the extension operator in (1-3). For simplicity, we assume that S is the paraboloid in $\mathbb{R}^n$ over $U = B_1$. Since the normal to the point $(\xi, \frac{1}{2}|\xi|^2)$ is $N(\xi) = \langle\xi\rangle^{-1}(-\xi, 1)$, the transversality condition in Theorem 1.3 is

$$|\det(N(\xi_1)\cdots N(\xi_n))| \sim_n \left|\det\begin{pmatrix} -\xi_1 & \cdots & -\xi_n \\ 1 & \cdots & 1 \end{pmatrix}\right|,$$

and the right hand side is the volume of the simplex spanned by the points ξ_k, hence transversal points avoid to lie in the same hyperplane.

We decompose $U \subset \mathbb{R}^{n-1}$ into caps α of radius $1/K_n$ and center c_α, for K_n a parameter to be fixed later. We estimate the size of the different terms $|\mathcal{E}(f\mathbb{1}_\alpha)(x)| = |\mathcal{E}f_\alpha(x)|$ at the point x. There are two possible scenarios:

(i) There are n transversal caps β_k with large contribution, *i.e.* the volume spanned by points in β_k is $\geq c\frac{1}{K_n^{n-1}}$ and $|\mathcal{E}f_{\beta_k}(x)| \geq \frac{1}{K_n^{n-1}}\sup_\alpha|\mathcal{E}f_\alpha(x)|$ (the latter term $\frac{1}{K_n^{n-1}}$ can be replaced by other $\frac{1}{K_n^C}$, but let us fix $C = n-1$). In this case

$$|\mathcal{E}f(x)| \leq \sum_\alpha |\mathcal{E}f_\alpha(x)| \leq K_n^{2(n-1)}|\prod_k \mathcal{E}f_{\beta_k}(x)|^{1/n}. \tag{1-18}$$

This seemingly crude inequality is not such a big loss, since K_n will be much smaller than R, the radius of the ball over which we integrate.

(ii) There are not transversal caps with large contribution. We want to prove that there exists a $(n-2)$-plane $V_{n-2} \subset \mathbb{R}^{n-1}$, such that for every α at distance $\geq C\frac{1}{K_n}$ from V_{n-2}, for $C \gg 1$, the inequality $|\mathcal{E}f_\alpha(x)| < \frac{1}{K_n^{n-1}}\sup_\alpha|\mathcal{E}f_\alpha(x)|$ holds.

Let $|\mathcal{E}f_{\alpha_1}(x)| = \sup_\alpha|\mathcal{E}f_\alpha(x)|$ and suppose that for every α at distance $\geq C\frac{1}{K_n}$ of α_1, we have that $|\mathcal{E}f_\alpha(x)| < \frac{1}{K_n^{n-1}}|\mathcal{E}f_{\alpha_1}(x)|$, then any plane V_{n-2} through α_1 will do. Otherwise, let α_2 be the furthest cap from α_1 such that $d(\alpha_1,\alpha_2) \geq C\frac{1}{K_n}$ and $|\mathcal{E}f_{\alpha_2}(x)| \geq \frac{1}{K_n^{n-1}}|\mathcal{E}f_{\alpha_1}(x)|$. Again, if for every α at distance $\geq C\frac{1}{K_n}$ from the line l joining α_1 and α_2 we have $|\mathcal{E}f_\alpha(x)| < \frac{1}{K_n^{n-1}}|\mathcal{E}f_{\alpha_1}(x)|$, then any plane V_{n-2} containing l will do. Otherwise, we continue the process choosing a furthest cap α_3 from the line l, until we reach $n-1$ caps α_i such that the $(n-2)$-volume spanned by them is $\geq C\frac{1}{K_n^{n-2}}$, so by our assumption on the lack of large contribution of transversal caps, necessarily for any cap α at distance $\geq C\frac{1}{K_n}$ from this $(n-2)$-plane we have $|\mathcal{E}f_\alpha(x)| < \frac{1}{K_n^{n-1}}|\mathcal{E}f_{\alpha_1}(x)|$.

In this case

$$\begin{aligned} |\mathcal{E}f(x)| &\leq |\sum_{d(\alpha,V_{n-2})<C/K_n} \mathcal{E}f_\alpha(x)| + \sum_{d(\alpha,V_{n-2})\geq C/K_n} |\mathcal{E}f_\alpha(x)|, \\ &\leq |\sum_{d(\alpha,V_{n-2})<C/K_n} \mathcal{E}f_\alpha(x)| + \sup_\alpha|\mathcal{E}f_\alpha(x)|. \end{aligned} \tag{1-19}$$

We observe, additionally, that the choice of V_{n-2} is stable at scale K_n, *i.e.* we can take V_{n-2} the same in a ball of radius K_n, because $|\mathcal{E}f_\alpha(x)|$ is roughly constant in balls B_{K_n} by uncertainty, as we observed previously in the multiscale analysis.

Summing the contributions from transversal and non-transversal terms, we have that

$$\begin{aligned}|\mathcal{E}f(x)| &\le K_n^{2(n-1)}|\prod_k \mathcal{E}f_{\beta_k}(x)|^{1/n} + |\sum_{d(\alpha,V_{n-2})<C/K_n} \mathcal{E}f_\alpha(x)| + \sup_\alpha |\mathcal{E}f_\alpha(x)|,\\ &= (1-20)+(1-21)+(1-22),\end{aligned}$$

where the choice of V_{n-2} and different caps β_k depends on x, but it is stable at scale K_n. We define now $K(R)$ as the best constant satisfying the inequality

$$\|\mathcal{E}f\|_{L^{p'}(B_R)} \le K(R)\|f\|_{p'}, \quad \text{for every } f,$$

so all we want is to show that $K(R) \lesssim R^\epsilon$.

We must control the $L^{p'}$-norm of the three terms, and we go first for the transversal term with the help of the multilinear Theorem 1.3 to get

$$\int_{B_R} |(1\text{-}20)|^{p'}\,dx \le K_n^{2p'(n-1)} \sum_{\substack{\alpha_1,\dots,\alpha_n\\ \text{transversal caps}}} \int_{B_R} |\prod_k \mathcal{E}f_{\alpha_k}(x)|^{p'/n}\,dx \le C_\epsilon K_n^C R^\epsilon \|f\|_{p'}^{p'},$$

the sum $\sum_{\alpha_1,\dots,\alpha_n}$ over all transversal caps is a rather crude estimation, but we can afford this loss; moreover, the constant C in K_n^C will not be important.

For the last term (1-22), we can estimate the $L^{p'}$-norm of every term $|\mathcal{E}f_\alpha(x)|$ exploiting the translation and dilatation symmetry of the paraboloid, carrying α to U, so that

$$\|\mathcal{E}f_\alpha\|_{L^{p'}(B_R)} \le K_n^{\frac{2n}{p'}-n+1} K(R/K_n)\|f_\alpha\|_{p'}.$$

This simple argument has been widely exploited in various contexts related to the restriction problem, see *e.g.* [68, 15]. Now,

$$\int_{B_R} |(1\text{-}22)|^{p'}\,dx \le \sum_\alpha \int_{B_R} |\mathcal{E}f_\alpha(x)|^{p'}\,dx \le K_n^{2n-p'(n-1)} K(R/K_n)^{p'}\|f\|_{p'}^{p'}.$$

The term involving the plane V_{n-2} will be our main concern, since it is just here where we will lose the sharp range $p' > \frac{2n}{n-1}$. For the time being, let us ignore this term and see how we could conclude the proof. From both estimations above we get

$$\|f\|_{L^{p'}(B_R)} \le [C_\epsilon K_n^C R^\epsilon + K_n^{\frac{2n}{p'}-(n-1)} K(R/K_n)]\|f\|_{p'},$$

which amounts to

$$K(R) \le C_\epsilon K_n^C R^\epsilon + K_n^{\frac{2n}{p'}-(n-1)} K(R/K_n).$$

Here we could iterate this bound, or one may see that $p' > \frac{2n}{n-1}$ implies that $\frac{2n}{p'} - (n-1) < 0$, so choosing K_n large enough we can assume that $K_n^{\frac{2n}{p'}-(n-1)} < \frac{1}{2}$; besides, it is not hard to see that $K(R/K_n) \leq K(R)$, hence for $K_n < R^\epsilon$ we get

$$K(R) \leq 2C_\epsilon K_n^C R^\epsilon \leq 2C_\epsilon R^{O(\epsilon)}, \qquad \text{for every } \epsilon > 0,$$

which is acceptable (recall that there is an ϵ-removal lemma), and proves the desired bound for $p' > \frac{2n}{n-1}$. We note by passing that, albeit not the original argument of Tao, Vargas and Vega, this method allows us to prove loosely that bilinear implies linear.

Coming back to the original problem, we cannot ignore the term (1-21) involving the hyperplane V_{n-2}. We divide U into caps $\alpha(K_{n-1})$ of radius $1/K_{n-1}$ for some parameter $K_{n-1} \ll K_n$ to be fixed later; this caps are unions of the caps $\alpha(K_n)$ in the first decomposition. We can think of the set

$$LV_{n-2} := \bigcup_{d(\alpha(K_n),V_{n-2})<C/K_n} \alpha(K_n), \tag{1-23}$$

as a "fat" paraboloid of width $\sim K_n^{-1}$. Therefore, we can apply the same dichotomy between transversality and non-transversality to "fat" caps $\alpha(K_{n-1}) \cap LV_{n-2} \neq \emptyset$, to get

$$\begin{aligned}|(1\text{-}21)| \leq{}& K_{n-1}^{2(n-2)} |\prod_{k=1}^{n-1} \mathcal{E} f_{\beta_k \cap LV_{n-2}}(x)|^{1/(n-1)} + \\ &+ \Big| \sum_{d(\alpha(K_{n-1})\cap LV_{n-2}, V_{n-3})<C/K_{n-1}} \mathcal{E} f_{\alpha(K_{n-1})\cap LV_{n-2}}(x) \Big| + \sup_{\alpha(K_{n-1})} |\mathcal{E} f_{\alpha(K_{n-1})\cap LV_{n-2}}(x)| \\ ={}& (1-24) + (1-25) + (1-26),\end{aligned}$$

where $\beta_k(K_{n-1}) \cap LV_{n-2}$ are transversal caps and $V_{n-3} \subset V_{n-2}$ is some $(n-3)$-plane depending on x; recall that the choice of V_{n-2} is stable at scale K_n, and now the choice of V_{n-3} is stable at scale $K_{n-1} \ll K_n$. As before, we must estimate the different contributions.

To estimate the term (1-26), we first remove the dependence on V_{n-2}

$$|\mathcal{E} f_{\alpha(K_{n-1})\cap LV_{n-2}}(x)| = |\mathcal{E} f_{\alpha(K_{n-1})} - \sum_{\substack{\alpha(K_n)\subset\alpha(K_{n-1}) \\ \alpha(K_n)\not\subseteq LV_{n-2}}} \mathcal{E} f_\alpha| \leq |\mathcal{E} f_{\alpha(K_{n-1})}(x)| + \sup_{\alpha(K_n)\subset U} |\mathcal{E} f_\alpha(x)|.$$

Hence, by dilatation we have

$$\begin{aligned}\int_{B_R} |(1\text{-}26)|^{p'}\, dx \leq{}& \sum_{\alpha(K_{n-1})} \int_{B_R} |\mathcal{E} f_\alpha(x)|^{p'}\, dx + \sum_{\alpha(K_n)} \int_{B_R} |\mathcal{E} f_\alpha(x)|^{p'}\, dx \\ \leq{}& K_{n-1}^{2n-p'(n-1)} K(R/K_{n-1})^{p'} + K_n^{2n-p'(n-1)} K(R/K_n)^{p'}] \|f\|_{p'}^{p'},\end{aligned}$$

and the contribution is acceptable for $p' > \frac{2n}{n-1}$, choosing K_{n-1} and K_n sufficiently large.

The transversal term (1-24) is harder to control than before and two estimations are now possible: the first estimation works in the range $p' \geq \frac{2(n-1)}{n-2}$ and the second in the range $\frac{2n}{n-1} < p' < \frac{2(n-1)}{n-2}$.

First estimation ($p' \geq 2\frac{n-1}{n-2}$). Bennett, Carbery and Tao [2, sec. 5] contemplated the possibility of lower levels of multilinearity, and they proved the following extension of Theorem 1.3.

Theorem 1.6. *If we are given $2 \leq m \leq n$ smooth compact hypersurfaces with boundary S_k, such that for every m points $\xi_k \in S_k$ the corresponding normal vectors $N(\xi_k)$ to S_k satisfy the transversality condition $|\det(N(\xi_1) \cdots N(\xi_m))| \geq \theta > 0$, then for $p' \geq \frac{2m}{m-1}$ and $p' \geq q\frac{m}{m-1}$ it holds*

$$\int_{B_R} \prod_{k=1}^{m} |\mathcal{E} f_k|^{p'/m} \, d\xi \leq C_\epsilon R^\epsilon \prod_{k=1}^{m} \|f_k\|_{q'}^{p'/m}, \tag{1-27}$$

for every $R \geq 1$ and $\epsilon > 0$.

Remark. The determinant is calculated choosing an orthogonal basis of the hyperplane spanned by the normal vectors.

We apply this theorem with $m = n-1$ to (1-24) to reach

$$\begin{aligned}\int_{B_R} |(1\text{-}24)|^{p'} \, dx &\leq K_{n-1}^{2(n-2)} \sum_{LV_{n-2}} \sum_{\substack{\alpha_k(K_{n-1}) \cap LV_{n-2} \neq \emptyset \\ \text{transversal caps}}} \int_{B_R} |\prod_{k=1}^{n-1} \mathcal{E} f_{\alpha_k \cap LV_{n-2}}(x)|^{p'/(n-1)} \, dx \\ &\lesssim_\epsilon (K_{n-1}K_n)^C R^\epsilon \|f\|_{p'}^{p'},\end{aligned} \tag{1-28}$$

which yields an acceptable contribution. The sum over LV_{n-2} is finite, since we are summing over hyperplanes defined by centers of caps $\alpha(K_n)$.

Second estimation ($\frac{2n}{n-1} < p' < \frac{2(n-1)}{n-2}$). To avoid technicalities, let us assume in (1-21) that we are summing over $\{\alpha(K_n) \mid \alpha \cap V_{n-2} \neq \emptyset\}$, instead of $\{\alpha(K_n) \mid d(\alpha(K_n), V_{n-2}) \leq C/K_n\}$. We assume further that $c_\alpha \in V_{n-2}$ and by translation symmetry we suppose that $V_{n-2} = \{x_{n-1} = 0\}$. We modify LV_{n-2} in (1-23) accordingly.

First, we try to get information at scale K_n, where by the uncertainty principle one can hardly distinguish a hyperplane from its $\frac{1}{K_n}$ neighborhood, so we estimate the average of $|\prod_k \mathcal{E} f_{\beta_k \cap LV_{n-2}}(x)|^{1/(n-1)}$ over balls B_{K_n}, which we denote by $\fint_B = \frac{1}{|B|}\int_B$. For simplicity, we suppose that the ball B_{K_n} is centered at the origin. Since the function $|\mathcal{E} f_{\beta_k(K_{n-1}) \cap LV_{n-2}}|$ is approximately constant in the variable x_{n-1} if we stay in B_{K_n}, we can integrate over sections $B_{K_n} \cap \mathbb{R}^{n-1}$ (recall that we assume that V_{n-2} is the hyperplane $\{x_{n-1} = 0\}$). Hence, by Hölder inequality and assuming that $\frac{2n}{n-1} < p' < \frac{2(n-1)}{n-2}$, we get

$$\begin{aligned}\fint_{B_{K_n}} |\prod_k \mathcal{E} f_{\beta_k \cap LV_{n-2}}(x)|^{\frac{p'}{n-1}} \, dx &\leq \Big(\fint_{B_{K_n}} |\prod_k \mathcal{E} f_{\beta_k \cap LV_{n-2}}(x)|^{\frac{2}{n-2}} \, dx\Big)^{\frac{p'(n-2)}{2(n-1)}} \\ &\approx \Big(\fint_{B_{K_n} \cap \mathbb{R}^{n-1}} |\prod_k \mathcal{E} f_{\beta_k \cap LV}(x', 0, x_n)|^{\frac{2}{n-2}} \, dx_1 \cdots \widehat{dx}_{n-1} dx_n\Big)^{\frac{p'(n-2)}{2(n-1)}}.\end{aligned}$$

Now, we are basically working with the extension operator in dimension $\mathbb{R}^{n-1}$. We repeat the approximation in (1-5) to write

$$\mathcal{E}f_{\beta(K_{n-1})\cap LV_{n-2}}(x',0,x_n) \approx \sum_{\alpha(K_n)\subset\beta(K_{n-1})\cap LV_{n-2}} \mathcal{E}f_\alpha(0)e(-\langle(x',0,x_n),\tilde{c}_\alpha\rangle),$$

that resembles the extension operator applied to the function

$$g_\beta = K_n^{n-2} \sum_{\alpha(K_n)\subset\beta(K_{n-1})} \mathcal{E}f_\alpha(0)\mathbb{1}_{\alpha\cap V_{n-2}},$$

defined in the paraboloid over V_{n-2}. We replace this in the inequality above, use the multilinear inequality (1-15) in dimension $n-1$, unfold the definition of g_β and use Hölder inequality to get

$$\begin{aligned}\fint_{B_{K_n}} |\prod_k \mathcal{E}f_{\beta_k\cap LV_{n-2}}(x)|^{\frac{p'}{n-1}}\,dx &\le \Big(\fint_{B_{K_n}\cap\mathbb{R}^{n-1}} |\prod_k \mathcal{E}g_{\beta_k}(x)|^{\frac{2}{n-2}}\,dx_1\cdots\widehat{dx}_{n-1}dx_n\Big)^{\frac{p'(n-2)}{2(n-1)}},\\ &\le C_\epsilon K_n^\epsilon\Big(\prod_k\sum_{\alpha\subset\beta_k}|\mathcal{E}f_\alpha(0)|^2\Big)^{\frac{p'}{2(n-1)}},\\ &\le C_\epsilon K_n^\epsilon\Big(\frac{K_n}{K_{n-1}}\Big)^{(n-2)(\frac{p'}{2}-1)}\sum_{\alpha(K_n)}|\mathcal{E}f_\alpha(0)|^{p'}. \end{aligned} \tag{1-29}$$

Once we have estimated the average over B_{K_n}, eliminating so the dependence on V_{n-2} in the right hand side of (1-29), we can sum over the balls B_{K_n} covering B_R to get

$$\int_{B_R} |(1\text{-}24)|^{p'}\,dx \le C_\epsilon K_{n-1}^C K_n^{\epsilon+(n-2)(\frac{p'}{2}-1)}\sum_{\alpha(K_n)}\int_{B_R}|\mathcal{E}f_\alpha(x)|^{p'}\,dx;$$

the last integrals can be estimated by dilatation of the caps $\alpha(K_n)$ to reach

$$\begin{aligned}\int_{B_R} |(1\text{-}24)|^{p'}\,dx &\le C_\epsilon K_{n-1}^C K_n^{\epsilon+(n-2)(\frac{p'}{2}-1)+2n-p'(n-1)}K(R/K_n)^{p'}\|f\|_{p'}^{p'},\\ &= C_\epsilon K_{n-1}^C K_n^{\epsilon+2+n-p'\frac{n}{2}}K(R/K_n)^{p'}\|f\|_{p'}^{p'}.\end{aligned}$$

The contribution is acceptable for $p' > 2\frac{n+2}{n}$ (the same exponent obtained with Tao's bilinear theorem), as long as we are in the range $\frac{2n}{n-1} < p' < \frac{2(n-1)}{n-2}$.

Both, first and second estimations, show that transversal terms in $(n-2)$-planes yield an acceptable contribution whenever

$$p' > 2\min(\frac{n-1}{n-2},\frac{n+2}{n}).$$

We estimate the term (1-25) involving V_{n-3} following the same pattern, until we reach 2-planes V_2, when there is no longer any non-trivial subspace of lower dimension. At each stage, we get the condition: the contribution of V_{m-1} terms is acceptable as long as

$$p' > 2\min(\frac{m}{m-1},\frac{2n-m+1}{2n-m-1}),\qquad \text{for } 1 < m < n.$$

All of this carries to the following range of boundedness for the extension operator:

$$p' \geq \begin{cases} 2\frac{4n+3}{4n-3} & \text{if } n = 0 \mod 3, \\ \frac{2n+1}{n-1} & \text{if } n = 1 \mod 3, \\ 4\frac{n+1}{2n-1} & \text{if } n = 2 \mod 3. \end{cases}$$

These ranges of exponents were the best result at that moment, and pushing the method even more, a slight improvement is possible; see [69].

Recently in [37, 34], Guth has pioneered the adaptation of the polynomial method to the restriction problem; see also [19, 58]. The philosophy of the method is to detect some kind of structure in the arrangement of tubes in the wave-packet decomposition. Essentially, he estimates $\int_{B_R} |\mathcal{E}f|^{p'}$ by partitioning B_R into cells $\{O_i\}$ by means of a carefully chosen polynomial P of controlled degree, *i.e.*, if $Z(P) = \{x \in B_R \mid P(x) = 0\}$, then $\cup_i O_i = B_R \setminus Z(P)$. He considers the contributions from $\sum_i \int_{O_i} |\mathcal{E}f|^{p'}$ and $\int_W |\mathcal{E}f|^{p'}$, where W is a $R^{-1/2+\epsilon}$-neighborhood of $Z(P)$. The contribution from the cells is usually easily controlled by an inductive argument, however if the main contribution comes from tubes "contained" in W (tubes crossing W transversely are not a very serious issue), then it indicates that the tubes in the wave-packet decomposition are structured and an inductive argument cannot handle its contribution. Hence, it is necessary to calculate the tangential term directly, observing that tubes passing through the same point are nearly coplanar and the L^4 norm can be estimated using Córdoba's argument, or bilinear methods, and the loss due to interpolation between L^2 and L^4 estimates prevented Guth from obtaining the sharp range of exponents. In [37], Guth got the best current bound for the paraboloid in $\mathbb{R}^3$, that is, $p' \geq 3.25$.

In [34], Guth studied lower levels of multilinearity. Theorem 1.6 does not involve any hypothesis about the curvature of the hypersurfaces, however when $m < n$ the curvature has some effect, as can be seen by comparing it with Tao's bilinear theorem. It is conjectured that for transversal subsets of the paraboloid in $\mathbb{R}^n$, the inequality

$$\|\prod_{i=1}^{m} \mathcal{E}f_i\|_{p'/m} \lesssim \prod_{i=1}^{m} \|f_i\|_2^{\frac{1}{m}}$$

holds for $p' \geq 2\frac{n+m}{n+m-2}$; Tao's bilinear theorem is $m = 2$ and Theorem 1.3 is $m = n$. Guth proved a weak analogue of the m-linear conjecture, that suffices in the Bourgain-Guth argument as a substitute of Theorem 1.6 in the first estimation, where we did not exploit the curvature of the paraboloid. After replacing the weak m-linear analogue in (1-28), the contribution of V_{m-1} terms is acceptable in the range:

$$p' > 2\min\left(\frac{n+m}{n+m-2}, \frac{2n-m+1}{2n-m-1}\right), \qquad \text{for } 1 < m < n.$$

This yields $L^{p'} \to L^{p'}$ bounds for the linear operator in the ranges

$$p' \geq \begin{cases} 2\frac{3n+2}{3n-2} & \text{if } n = 0 \mod 2, \\ 2\frac{3n+1}{3n-3} & \text{if } n = 1 \mod 2; \end{cases}$$

which is the best result to date for $n \geq 4$.

1.3 Some Applications

The restriction theory has many applications, and they are not limited in scope to the study of summation of Fourier integrals or partial differential equations. In chapter 3 we will discuss an application to Falconer's conjecture, a problem in geometric measure theory. But we cannot leave without mentioning the remarkable application of the decoupling theory, a spin-off from the restriction problem, to number theory, in particular, to the resolution of the Vinogradov's mean value conjecture.

We already mentioned the square function conjecture, that asserts that for a decomposition of $\mathcal{E}f$ into caps α of radius δ, the inequality

$$\|\mathcal{E}f\|_{L^{p'}(B_{\delta-2})} \leq C\|\big(\sum_{\alpha}|\mathcal{E}f_\alpha|^2\big)^{\frac{1}{2}}\|_{L^{p'}(B_{\delta-2})}$$

holds in the range $2 \leq p' < \frac{2n}{n-1}$. The conjecture has been verified only for $\mathbb{R}^2$, but seems completely out of reach in general. Wolff [74] observed that the weaker inequality

$$\|\mathcal{E}f\|_{L^{p'}(B_{\delta-2})} \leq C\big(\sum_{\alpha}\|\mathcal{E}f_\alpha\|^2_{L^{p'}(B_{\delta-2})}\big)^{\frac{1}{2}} \qquad \text{(weaker because } p' \geq 2), \tag{1-30}$$

works sufficiently well for some applications, instead of the full square function conjecture. We had to wait for the development of the multilinear theory, to see the most incredible progress towards the weak square function conjecture.

The analogue of the Strichartz inequality for the Schrödinger equation in the torus is

$$\|\sum_{|m|\leq N} a_n e(-\langle m, x\rangle - x_n|m|^2)\|_{L^{2\frac{n+1}{n-1}}(\mathbb{T}^n)} \leq C\|a\|_{\ell^2}.$$

Although the Euclidean case had been solved in the seventies, the discrete analogue seemed intractable, and the sole progress had been done by Bourgain [5], using advanced tools from analytic number theory. Many years later, Bourgain and Demeter in [11] (see also [14]) used the multilinear theory to prove (1-30), up to logarithmic losses, in the sharp range $2 \leq p' \leq 2\frac{n+1}{n-1}$ for the paraboloid (notice that the range is wider than the one for the square function conjecture). By uncertainty and some additional considerations, this has as a consequence the discrete Strichartz inequality for the paraboloid, surprisingly, without using any number theory. This motivated them to research even further on weak square functions or decoupling inequalities, as we now know them, for other manifolds, see e.g. [12, 13, 10].

One of the biggest achievements came with the resolution of the Vinogradov's mean value conjecture. An important problem in analytic number theory is to bound the quantity

$$J_{s,k}(X) = \int_{\mathbb{T}^k} |\sum_{1\leq t\leq X} e(x_1 t + \cdots + x_k t^k)|^{2s}\, dx;$$

more specifically, it was conjectured for a long time that $J_{s,k}(X) \lesssim_\epsilon (X^{s+\epsilon} + X^{2s-\frac{1}{2}k(k+1)+\epsilon})$. In spite of the large body of research, using analytic number theory, the conjecture stood firm for $k > 3$. Bourgain, Demeter and Guth [15] were able to prove the decoupling inequality associated to the moment curve $(t, \dots, t^k)$, and with it at hand the Vinogradov's mean value conjecture was a mere technical matter, but all the more impressive, no number theory is needed in the proof. For additional information, see [59].

1.4 What does this thesis contain?

All the preceding discussion should show the relevance of multilinear inequalities in the progress of the restriction theory. The core of this thesis revolves around this subject.

As we said, it is unknown, in general, if the sharp constant in the Kakeya multilinear inequality, Theorem 1.5, carries over to the multilinear inequality, Theorem 1.3. However, during the thesis I proved the trilinear inequality for the hyperbolic paraboloid with sharp dependence on transversality.

Theorem 1.7. *If we are given three subsets S_k of the truncated hyperbolic paraboloid $S = \{(\xi_1, \xi_2, \xi_1\xi_2) \mid \xi_1^2 + \xi_2^2 \leq 1\}$ in $\mathbb{R}^3$, such that for every three points $\xi_k \in S_k$ the corresponding normal vectors $N(\xi_k)$ to S_k satisfy the transversality condition $|\det(N(\xi_1)\, N(\xi_3)\, N(\xi_3))| \sim \theta > 0$, then*

$$\int_{B_R} \prod_{k=1}^{3} |\mathcal{E} f_k| \, d\xi \leq C_\epsilon \theta^{-\frac{1}{2}} R^\epsilon \prod_{k=1}^{3} \|f_k\|_2, \tag{1-31}$$

for every $R \geq 1$ and $\epsilon > 0$. The constant C_ϵ does not depend on R or θ.

This is the main result in Chapter 2. This theorem extends the work of Ramos [60] about elliptic surfaces in $\mathbb{R}^3$, who proved the next theorem.

Theorem 1.8. *If we are given three subsets S_k of the truncated paraboloid in $\mathbb{R}^3$, such that for every three points $\xi_k \in S_k$ the corresponding normal vectors $N(\xi_k)$ to S_k satisfy the transversality condition $|\det(N(\xi_1)\, N(\xi_3)\, N(\xi_3))| \sim \theta > 0$, then*

$$\int_{B_R} \prod_{k=1}^{3} |\mathcal{E} f_k| \, d\xi \leq C_\epsilon \theta^{-\frac{1}{2}} R^\epsilon \prod_{k=1}^{3} \|f_k\|_2, \tag{1-32}$$

for every $R \geq 1$ and $\epsilon > 0$. The constant C_ϵ does not depend on R or θ.

Additionally, I wish to discuss in Chapter 4 the possibility of extending Ramos' theorem to higher dimensions.

In chapter 3 I describe a brief result about the average decay of the Fourier transform of compactly supported measures, which is related to Falconer's conjecture.

This thesis resulted in two papers:

- A Trilinear Restriction Estimate for the Hyperbolic Paraboloid with Sharp dependence on Transversality, submitted.
- Examples of Measures with Slow Decay of the Spherical Means of the Fourier Transform, *Proc. Amer. Math. Soc.*, **146**, 2617–2621.

2 Sharp Depencende on Transversality for the Hyperbolic Paraboloid

We already mentioned the contribution of Ramos in [60] to the improvement of the dependence on transversality in Theorem 1.3, when the surface S has positive curvature. In this chapter we extend this result to the hyperbolic paraboloid, a model phase with negative curvature. Let us outline the proof of Ramos and how we modify it in the case of the hyperbolic paraboloid.

The proof of Theorem 1.3 is by multiscale analysis, similar to Bourgain's method in section 1.1, so we use the same notation there. One makes a wave-packet decomposition of $U \subset \mathbb{R}^{n-1}$ by using caps α_k of radius δ, so that we can write $\mathcal{E} f_k = \sum_{\alpha_k} \mathcal{E}(f_{k,\alpha_k})$ for $k = 1, \ldots, n$. In a ball of radius δ^{-2}, each function $\mathcal{E}(f_{k,\alpha_k})$ looks like a collection of modulated tubes of radius δ^{-1} pointing in the direction of the normal vector at the center of $\tilde{\alpha}_k$. If n tubes intersect, one from each hypersurface $\tilde{U}_k = S_k$, then the best we can say is that their intersection contains a ball of radius δ^{-1}. Using the principle of uncertainty and the multilinear Kakeya inequality, we can exploit local averages at scale δ^{-1} to get a global bound at scale δ^{-2}. Although this process is efficient regarding the dependence on δ, it yields poor results about transversality. One of the key observations in [60] is the necessity of averaging over "optimal" parallelepipeds P, not over balls $B_{\delta^{-1}}$. If we choose the caps $\alpha_k \subset U_k$ carefully, instead of bare disks of radius δ, then the functions $f_k = \mathbb{1}_{\alpha_k}$ attain the optimal dependence on transversality; moreover, the functions $\mathcal{E}\mathbb{1}_{\alpha_k}$ look like tubes with elliptic section, and n of these tubes from each S_k intersect in an "optimal" parallelepiped P, over which the multiscale analysis is efficient. The caps needed for the paraboloid and hyperbolic paraboloid are different, but they can be found in a systematic way; see Chapter 4.

As every inductive argument, one needs a starting point. Bennett, Carbery and Tao used a trivial bound of the extension operator and integrated over B_1, which suffices in their argument. However, this first step is insufficient for us, because our initial decomposition into caps α_k already contains many pieces and we must paste them efficiently to integrate over the parallelepiped P. To overcome this obstacle Ramos used a refinement of Córdoba's L^4 orthogonality, but seeking an improvement upon the linear restriction problem, he used a strong property of orthogonality. We use a weaker property of orthogonality to prove the sharp inequality for the hyperbolic paraboloid, and defer the stronger, and harder to prove, orthogonality to the last section, since we do not need it for our purposes.

The improved multiscale analysis and orthogonality do not work over a general triplet of sets $\{S_k\}_{k=1,2,3}$, but over especially constructed triplets. Hence, to finish the proof, we cover efficiently an arbitrary triplet of sets with especial triplets. We cannot use the same triplets defined by Ramos, since we must avoid certain configurations of sets, as noted by Vargas [71] and Lee [48]. Therefore, we adapt the pairs of sets used by Vargas to the trilinear setting, exploiting the methods of Ramos.

2.1 Preliminary Reductions

The condition of transversality in Theorem 1.7 between surfaces S_k in the hyperbolic paraboloid translates to a condition on the area of the triangle spanned by triplets of points $\xi_k = (\xi_{k,1}, \xi_{k,2}) \in U_k$; in fact, since the normal at $(\xi, \xi_1\xi_2)$ is $(\xi_2, \xi_1, -1)$, the condition of transversality for three points $\xi_k \in S_k$ is

$$|\det \begin{pmatrix} \xi_{1,2} & \xi_{2,2} & \xi_{3,2} \\ \xi_{1,1} & \xi_{2,1} & \xi_{3,1} \\ -1 & -1 & -1 \end{pmatrix}| = \frac{1}{2}\text{area}(\xi_1, \xi_2, \xi_3) \sim \theta.$$

If three surfaces satisfy the transversality conditions in Theorem 1.7 with parameter $\theta > 0$, then we say that they are θ-transverse.

2.1.1 Coarse Decomposition

Let S_1, S_2 and S_3 be three θ-transverse compact subsets of the hyperbolic paraboloid, with corresponding projections U_1, U_2 and U_3. We will present in this section a long series of decompositions of $U_1 \times U_2 \times U_3$, aimed to ensure suitable properties of separation between the sets and orthogonality between the functions supported on them. In order to achieve the latter, we adapt the Whitney type decomposition used by Vargas [71] to the trilinear setting.

Before starting, recall the trivial estimates $|\mathcal{E}f| \leq |\text{supp}\, f|^{\frac{1}{2}} \|f\|_2$ and

$$\int_{B_R} |\mathcal{E}f_1 \mathcal{E}f_2 \mathcal{E}f_3|\, dx \leq CR^3 \prod_k |\text{supp}\, f_k|^{\frac{1}{2}} \|f_k\|_2; \tag{2-1}$$

hence, the sharp trilinear bound holds for $R^3 \leq \theta^{-1/2}$, so we assume that $R^3 > \theta^{-1/2}$.

We describe first a decomposition of $U_1 \times U_2$ into separated rectangles. Consider all the intervals $(l2^{-i}, (l+1)2^{-i}) \in [-1,1]$, for $l \in \mathbb{Z}$ and $i \in \{1, \ldots, M\}$, where M is some integer to be fixed later. We say that two intervals I_1^i and I_2^i of length 2^{-i} are close if they are not adjacent, but their parents are. Notice that the set $[-1,1] \times [-1,1]$ is covered by products $I_1^i \times I_2^i$ of close intervals, plus some products $I_1^M \times I_2^M$ of small pairs of adjacent or identical intervals, as in Figure **2-1**. We define now rectangles $\tau_k^{i,j} = I_k^i \times I_k^j$ for $k = 1$ and 2, and we say that two rectangles $\tau_1^{i,j}$ and $\tau_2^{i,j}$ are close if I_1^i is close to I_2^i and I_1^j is close

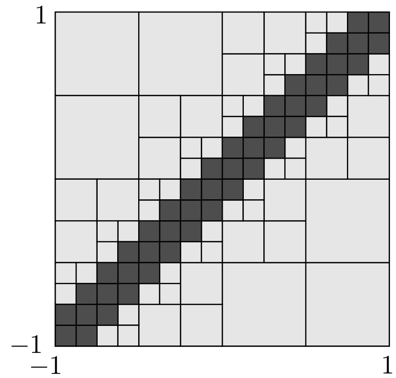

Figure **2-1**: The square $[-1,1] \times [-1,1]$ is decomposed up to scale $M = 3$. The pale gray squares represent $I_1^i \times I_2^i$ for pairs of close intervals. The dark gray squares are products of pairs of adjacent or identical intervals with side length 2^{-M}.

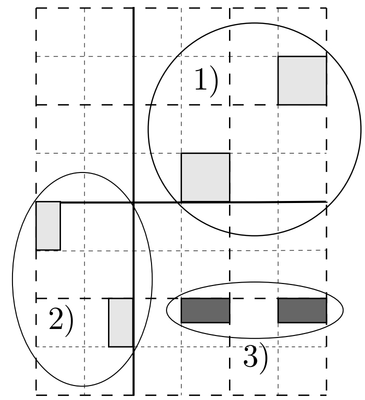

Figure **2-2**: Different possible pairs of rectangles, enclosed by ovals for clarity. The first two pairs are examples of close rectangles. The third pair is not close, which may happen if at least one side has length 2^{-M}.

to I_2^j; this relationship between rectangles will be denoted by $\tau_1^{i,j} \sim \tau_2^{i,j}$. By the above, we can decompose $[-1,1]^2 \times [-1,1]^2$ as a union of pairwise disjoint sets $\tau_1^{i,j} \times \tau_2^{i,j}$, such that $\tau_1^{i,j} \sim \tau_2^{i,j}$, plus some products of small adjacent rectangles $\tau_1^{i,j} \times \tau_2^{i,j}$, for $i = M$ or $j = M$, where some pair of intervals are adjacent or identical; see Figure **2-2**. These pairs of small rectangles for $i = M$ or $j = M$ do not enjoy, in general, suitable properties of separation, so later in this section we will deal with this case separately. It is important to remark that close rectangles are not crossed by a common vertical or horizontal line.

For each set $\tau_1^{i,j} \times \tau_2^{i,j}$ formed by rectangles $\tau_1^{i,j} \sim \tau_2^{i,j}$, for $i, j \leq M$, our next task is to decompose $U_3 \subset [-1,1]^2$ so that $\tau_1^{i,j} \times \tau_2^{i,j} \times U_3$ can be written as a disjoint union of sets $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3$, where τ_3 are rectangles of controlled size and distance from the pair of rectangles in $\tau_1^{i,j} \times \tau_2^{i,j}$. We choose thus a set $\tau_1^{i,j} \times \tau_2^{i,j}$ and apply the dilation $(\xi_1, \xi_2) \in \mathbb{R}^2 \mapsto (2^i \xi_1, 2^j \xi_2)$ to each rectangle, transforming the set $\tau_1^{i,j} \times \tau_2^{i,j}$ into a product $Q_1 \times Q_2$ of a pair of unit squares at distance ~ 1 between each other; also, $[-1,1]^2$ is transformed into $[-2^i, 2^i] \times [-2^j, 2^j]$. Now, we will make a Whitney decomposition of $\mathbb{R}^2$ into squares Q_3, so that the distance between the centers of Q_3 and Q_k, for $k = 1$ and 2, is comparable to the side length of Q_3.

Suppose that the rectangle $\tau_k^{i,j} = I_k^i \times I_k^j$, for $k = 1, 2$, lies in the parent rectangle $\tau_k^{i-1,j-1} = I_k^{i-1} \times I_k^{j-1}$, which is transformed into a square Q'_k of side length 2 containing Q_k. We translate both squares Q'_1 and Q'_2, say by a vector v, so that they have one of the following configurations:

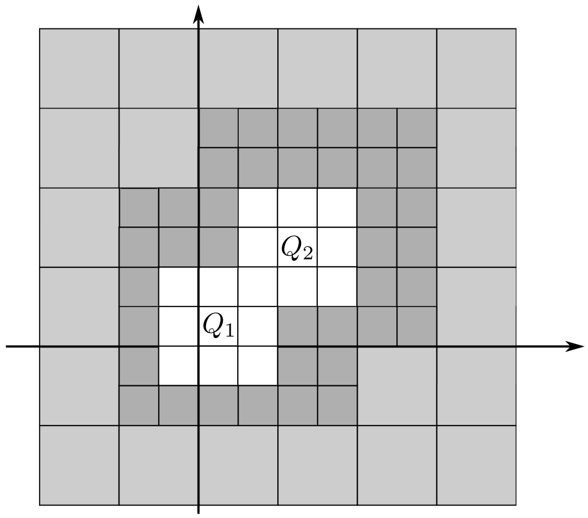

Figure **2-3**: The unit cubes in white are adjacent to (Q_1, Q_2), including Q_1 and Q_2 themselves. The shaded cubes of side length 1 and 2 are close to (Q_1, Q_2).

- $v + Q'_{k_1} = [0,2]^2$ and $v + Q'_{k_2} = [2,4]^2$;
- $v + Q'_{k_1} = [0,2] \times [2,4]$ and $v + Q'_{k_2} = [2,4] \times [0,2]$.

Let us drop the vector v, since its specific value is not important. Now, we consider dyadic squares $Q_3^r = (l_1 2^r, (l_1+1)2^r) \times (l_2 2^r, (l_2+1)2^r)$, for $l_j \in \mathbb{Z}$ and $r \geq 0$; notice that the unit squares Q_1 and Q_2 have the form $(l_1, l_1+1) \times (l_2, l_2+1)$ for some integers $0 \leq l_1, l_2 < 4$. Now, we say that two squares are close if they are not adjacent, but their parents are. We say that a unit square Q_3^0 is close to the pair (Q_1, Q_2) if it is close to one of the squares Q_1 or Q_2, but not adjacent to the other; otherwise, we say that Q_3^0 is adjacent to (Q_1, Q_2) if it is adjacent or equal to some of the squares Q_1 or Q_2; see Figure **2-3**. We write $Q_3^0 \sim (Q_1, Q_2)$ to denote that Q_3^0 is close to the pair (Q_1, Q_2), and $Q_3^0 \nsim (Q_1, Q_2)$ when Q_3^0 is adjacent.

The union of the squares $Q_3^0 \sim (Q_1, Q_2)$, which lie at a distance ~ 1 from Q_1 and Q_2, and the squares adjacent to (Q_1, Q_2) is a partial covering of the plane, which we call Ω_0. We define now inductively an increasing sequence of sets Ω_r, for $r \geq 0$, until we cover the translated rectangle $v + [-2^i, 2^i] \times [-2^j, 2^j]$. To do that, we say that Q_3^r is close to the pair (Q_1, Q_2) if it is not adjacent to $[0, 2^r]^2$, but its parent is to $[0, 2^{r+1}]^2$, and additionally $Q_3^r \cap \Omega_{r-1} = \emptyset$.[1] The set Ω_r is simply the union of Ω_{r-1} and the squares $Q_3^r \sim (Q_1, Q_2)$. We do not have to bother about adjacent squares Q_3^r in this case; see Figure **2-3** for the partial covering Ω_1. Restore the squares Q_1, Q_2 to their original position, carrying with them the corresponding squares Q_3^r, such that we get the disjoint covering

$$Q_1 \times Q_2 \times [-2^i, 2^i] \times [-2^j, 2^j] \subset \bigcup_{\substack{Q_3^0 \nsim (Q_1,Q_2) \\ Q_3^r \sim (Q_1,Q_2), \\ 0 \leq r \leq M}} Q_1 \times Q_2 \times Q_3^r.$$

[1] Actually, this last condition is only relevant when $r = 1$.

Finally, we scale back the triplets $Q_1 \times Q_2 \times Q_3^r$ to $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i-r,j-r}$, where $\tau_3^{i-r,j-r}$ are rectangles of dimensions $2^{r-i} \times 2^{r-j}$.

We repeat the previous process with all the pairs $\tau_1^{i,j} \times \tau_2^{i,j}$, until we get a pairwise disjoint covering

$$[-1,1]^2 \times [-1,1]^2 \times [-1,1]^2 \subset \Big(\bigcup_{\substack{\tau_1^{i,j} \sim \tau_2^{i,j} \\ 0 \le i,j \le M}} \bigcup_{\substack{\tau_3^{i,j} \nsim (\tau_1^{i,j}, \tau_2^{i,j}) \\ \tau_3^{i-r,j-r} \sim (\tau_1^{i,j}, \tau_2^{i,j}), \\ 0 \le r \le M}} \tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i-r,j-r} \Big) \bigcup \text{Residue}. \quad (2\text{-}2)$$

The set Residue is the union of the sets $\tau_1^{i,j} \times \tau_2^{i,j} \times [-1,1]^2$, where $\tau_1^{i,j}$ and $\tau_2^{i,j}$ are adjacent, which may happen when $i = M$ or $j = M$. We denote by $\mathcal{A}$ the collection of all the sets $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i-r,j-r}$ in the above disjoint covering, plus the negligible triplets $\tau_1^{i,j} \times \tau_2^{i,j} \times [-1,1]^2$ in the set Residue, which we omit most of the time. For the time being, we do not need to distinguish between close and adjacent rectangles $\tau_3^{i,j}$.

Suppose we have proved the trilinear estimate

$$\int_{B_R} |\prod_{k=1}^{3} \mathcal{E} f_k| \, dx \le L(R) \prod_{k=1}^{3} \|f_k\|_2, \quad (2\text{-}3)$$

for triplets of functions supported inside sets $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i-r,j-r} \in \mathcal{A}$, as in (2-2). By the triangle inequality, the bound above and Cauchy-Schwarz we get

$$\begin{aligned} \int_{B_R} |\mathcal{E} f_1 \mathcal{E} f_2 \mathcal{E} f_3| \, dx &\le \sum_{\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i-r,j-r} \in \mathcal{A}} \int_{B_R} |\mathcal{E}(f_1 \mathbb{1}_{\tau_1^{i,j}}) \mathcal{E}(f_2 \mathbb{1}_{\tau_2^{i,j}}) \mathcal{E}(f_3 \mathbb{1}_{\tau_3^{i-r,j-r}})| \, dx \\ &\le L(R) \sum_{\mathcal{A}} \|f_1 \mathbb{1}_{\tau_1^{i,j}}\|_2 \|f_2 \mathbb{1}_{\tau_2^{i,j}}\|_2 \|f_3 \mathbb{1}_{\tau_3^{i-r,j-r}}\|_2 \\ &\le L(R) \sup_{\mathcal{A}} \|f_3 \mathbb{1}_{\tau_3^{i-r,j-r}}\|_2 \big(\|f_1 \Sigma_{\mathcal{A}} \mathbb{1}_{\tau_1^{i,j}}\|_2^2 \big)^{\frac{1}{2}} \big(\|f_2 \Sigma_{\mathcal{A}} \mathbb{1}_{\tau_2^{i,j}}\|_2^2 \big)^{\frac{1}{2}}. \end{aligned}$$

It is clear that $\sup_{\mathcal{A}} \|f_3 \mathbb{1}_{\tau_3^{i-r,j-r}}\|_2 \le \|f_3\|_2$ and we need to get a bound of $\sum_{\mathcal{A}} \mathbb{1}_{\tau_k^{i,j}}$. A point $\xi \in [-1,1]^2$ lies in at most M^2 rectangles $\tau_k^{i,j}$, and each rectangle is related to $\lesssim 1$ close rectangles. In turn, each pair $(\tau_1^{i,j}, \tau_2^{i,j})$ is close to $\lesssim M$ rectangles $\tau_3^{i-r,j-r}$, hence after collecting all the contributions we conclude that $\sum_{\mathcal{A}} \mathbb{1}_{\tau_k^{i,j}} \lesssim M^3$. Therefore,

$$\int_{B_R} |\mathcal{E} f_1 \mathcal{E} f_2 \mathcal{E} f_3| \, dx \le C M^6 L(R) \prod_k \|f_k\|_2. \quad (2\text{-}4)$$

It remains to choose M. Recall that at the beginning, when we decomposed $U_1 \times U_2$ using rectangles $\tau_k^{i,j} = I_k^i \times I_k^j$, we warned that when $i = M$ or $j = M$ we possibly do not have good properties of separation; however, by using the trivial estimate (2-1) we have $\|\prod_k \mathcal{E} f_k\|_{L^1(B_R)} \le C R^3 2^{-M} \prod_k \|f_k\|_2$, so if we fix M so large that $2^{-M} \sim R^{-3} \theta^{-1/2}$ or $M \sim \log(R^3 \theta^{1/2}) \lesssim \log R$, then the loss due to M is acceptable and for these triplets $L(R) \le$

$C\theta^{-\frac{1}{2}}$. Hence, to prove Theorem 1.7 it suffices to prove the trilinear inequality (2-3) with $L(R) \leq C(\log R)^C\theta^{-\frac{1}{2}}$ for sets $U_1 \times U_2 \times U_3 \subset \tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i-r,j-r} \in \mathcal{A}$, when $\tau_1^{i,j} \sim \tau_2^{i,j}$.

The sets $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i-r,j-r}$, for $\tau_3^{i-r,j-r} \sim (\tau_1^{i,j}, \tau_2^{i,j})$, already enjoy good properties of separation between rectangles, which we will exploit in the next subsection, hence we are left with the case when $\tau_3^{i,j} \nsim (\tau_1^{i,j}, \tau_2^{i,j})$. Recall that $\tau_3^{i,j}$ is adjacent to $(\tau_1^{i,j}, \tau_2^{i,j})$ if it is adjacent or equal to at least one of the rectangles $\tau_1^{i,j}$ or $\tau_2^{i,j}$. Our next task is to decompose the sets $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i,j}$, for $\tau_3^{i,j} \nsim (\tau_1^{i,j}, \tau_2^{i,j})$, into sets with good properties of separation, but we already know how to do that, so basically we will repeat the previous process of reduction with $U_1 \times U_2 \times U_3$ replaced by $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i,j}$.

First, we enforce some separation by dividing all the rectangles $\tau_k^{i,j}$, for $k = 1, 2, 3$, into four similar rectangles, yielding the decomposition $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i,j} = \bigcup \tau_1^{i+1,j+1} \times \tau_2^{i+1,j+1} \times \tau_3^{i+1,j+1}$; the increment in the number of sets can be controlled in (2-3) by the triangle inequality. For each set $\tau_1^{i+1,j+1} \times \tau_2^{i+1,j+1} \times \tau_3^{i+1,j+1}$, the rectangle $\tau_3^{i+1,j+1}$ is closest in distance or equal to some of the remaining two rectangles,[2] say $\tau_1^{i+1,j+1}$, while it will be necessarily separated from the other, in this case $\tau_2^{i+1,j+1}$. If we repeat all the process described above, but assuming that $U_1 \times U_2 \times U_3 \subset \tau_1^{i+1,j+1} \times \tau_2^{i+1,j+1} \times \tau_3^{i+1,j+1}$ and replacing from the beginning $U_1 \times U_2$ by $U_1 \times U_3$, and subsequently U_3 by U_2, then at the end we get a disjoint covering similar to (2-2), but permuting the indices 2 and 3. Since $\tau_2^{i+1,j+1}$ is already separated from $\tau_1^{i+1,j+1}$ and $\tau_3^{i+1,j+1}$, the rectangles $\tau_2^{i',j'} \nsim (\tau_1^{i',j'}, \tau_3^{i',j'})$ disappear from the decomposition. Repeat the chain of arguments from (2-3) to (2-4), permuting 2 and 3, to conclude that to prove Theorem 1.7 for sets $U_1 \times U_2 \times U_3 \subset \tau_1^{i+1,j+1} \times \tau_2^{i+1,j+1} \times \tau_3^{i+1,j+1}$, it suffices to prove (2-3) with $L(R) \leq C(\log R)^C\theta^{-\frac{1}{2}}$ for triplets of sets inside $\tau_1^{i,j} \times \tau_2^{i-r,j-r} \times \tau_3^{i,j}$, where $\tau_1^{i,j} \sim \tau_3^{i,j}$ and $\tau_2^{i-r,j-r} \sim (\tau_1^{i,j}, \tau_3^{i,j})$. The case $\tau_3^{i+1,j+1}$ closest in distance to $\tau_2^{i+1,j+1}$ is handled similarly. We get thus the following Lemma.

Lemma 2.1. *The trilinear inequality in Theorem 1.7 is a consequence of the inequality*

$$\int_{B_R} \Big|\prod_{k=1}^{3} \mathcal{E}f_k\Big|\, dx \leq C(\log R)^C\theta^{-1/2}\prod_{k=1}^{3}\|f_k\|_2, \tag{2-5}$$

where the functions f_{k_1}, f_{k_2} and f_{k_3} are supported inside sets $\tau_{k_1}^{i,j}$, $\tau_{k_2}^{i,j}$ and $\tau_{k_3}^{i-r,j-r}$ respectively, such that $\tau_{k_1}^{i,j} \sim \tau_{k_2}^{i,j}$, $\tau_{k_3}^{i-r,j-r} \sim (\tau_{k_1}^{i,j}, \tau_{k_2}^{i,j})$, and (k_1, k_2, k_3) is a permutation of $(1, 2, 3)$.

For definiteness, let us suppose that $U_1 \times U_2 \times U_3$ is contained in a set $\tau_1^{i,j} \times \tau_2^{i,j} \times \tau_3^{i-r,j-r}$, for $\tau_1^{i,j} \sim \tau_2^{i,j}$ and $\tau_3^{i-r,j-r} \sim (\tau_1^{i,j}, \tau_2^{i,j})$. We prefer now to work with a normalized triplet. We apply the transformation $T : (\xi_1, \xi_2) \mapsto (2^{i-r}\xi_1, 2^{j-r}\xi_2)$, sending $\tau_1^{i,j}$ and $\tau_2^{i,j}$ to squares $\tau_1^{r,r}$ and $\tau_2^{r,r}$ of side length 2^{-r}, and $\tau_3^{i-r,j-r}$ to a unit square $\tau_3^{1,1}$. By the translation symmetry of the extension operator, we assume that the center of $\tau_1^{r,r}$ coincides with the origin. The action of the affine transformation T over a function f, sending f to $f_T = f \circ T^{-1}$, has as

[2] In case of ambiguity, choose any rectangle.

consequence in the extension operator

$$\begin{aligned}\mathcal{E}f(x) &= \int_U f_T(T\xi)e(-\langle x',\xi\rangle - x_3\xi_1\xi_2)\,d\xi \\ &= \frac{1}{|\det T|}\int_{TU} f_T(\xi)e(-\langle x',T^{-1}\xi\rangle - x_3\varphi(T^{-1}\xi))\,d\xi \\ &= 2^{2r-i-j}\mathcal{E}f_T(2^{r-i}x_1, 2^{r-j}x_2, 2^{2r-i-j}x_3).\end{aligned}$$

We apply this transformation to each function f_k in the trilinear expression to get

$$\begin{aligned}\|\prod_k \mathcal{E}f_k\|_{L^1(B_R)} &= 2^{3(2r-i-j)}\|\prod_k \mathcal{E}f_{k,T}(2^{r-i}x_1, 2^{r-j}x_2, 2^{2r-i-j}x_3)\|_{L^1(B_R)} \\ &\leq 2^{2r-i-j}\|\prod_k \mathcal{E}f_{k,T}(x_1,x_2,x_3)\|_{L^1(B_{2^M R})}.\end{aligned}$$

If the area spanned by points $\xi_k \in U_k$ is $\sim \theta$, then when they are scaled by T the new area is $\sim \tilde{\theta} = 2^{i+j-2r}\theta$, hence if we assume the sharp trilinear inequality as proved for functions supported inside sets of the form $\tau_1^{r,r} \times \tau_2^{r,r} \times \tau_3^{1,1}$, then we get

$$\begin{aligned}\|\prod_k \mathcal{E}f_k\|_{L^1(B_R)} &\leq C2^{2r-i-j}(\log(2^M R))^C\tilde{\theta}^{-1/2}\prod_k \|f_{k,T}\|_2 \\ &\leq C(2^{i+j-2r})^{1/2}(\log R)^C\tilde{\theta}^{-1/2}\prod_k \|f_k\|_2 \\ &= C(\log R)^C\theta^{-1/2}\prod_k \|f_k\|_2;\end{aligned}$$

we used here that $M \lesssim \log R$. Hence, after Lemma 2.1, it suffices to assume the case $\tau_{k_1}^{r,r}$, $\tau_{k_2}^{r,r}$ and $\tau_{k_3}^{1,1}$.

2.1.2 Fine Decomposition

By the end of the previous subsection, we concluded that it suffices to assume that the set $U_1 \times U_2 \times U_3$ is a subset of a triplet $\tau_1^{j,j} \times \tau_2^{j,j} \times \tau_3^{1,1}$, where $\tau_1^{j,j}$ and $\tau_2^{j,j}$ are close squares of side length 2^{-j}, and τ_3 is a unit square[3] at distance ~ 1 from $\tau_1^{j,j}$ and $\tau_2^{j,j}$. We apply the symmetry $\xi \mapsto (-\xi_1, \xi_2)$, if necessary, to assume that the line l joining the centers of $\tau_1^{j,j}$ and $\tau_2^{j,j}$ has positive slope, as in Figure **2-3**. Furthermore, if $l^\perp$ is perpendicular to l an pass through the center of $\tau_1^{j,j}$, then by applying the transformation $\xi \mapsto (-\xi_1, -\xi_2)$ we can assume that τ_3 intersects the upper-right half plane defined by $l^\perp$.

We exploit now the properties of separation to get a finer decomposition of $U_1 \times U_2 \times U_3$. Since the surfaces S_1, S_2 and S_3 are θ-transverse, the area of the triangle spanned by any triplet of points $\xi_k \in U_k$ is $\sim \theta$. Given two points $\xi_1 \in U_1$ and $\xi_2 \in U_2$, let $l(\xi_1, \xi_2)$ be the line joining them. Suppose that the acute angle spanned by the lines $l(\xi_1, \xi_2)$ and $l(\xi_1, \xi_3)$ for some

[3] We have dropped the superscript from $\tau_3^{1,1}$.

$\xi_3 \in U_3$ is ϕ, thus the area of the triangle with vertices ξ_1, ξ_2, ξ_3 is $A = \frac{1}{2}|\xi_2 - \xi_1||\xi_3 - \xi_1| \sin\phi$, and by the properties of separation between points, that is, $|\xi_2 - \xi_1| \sim 2^{-j}$ and $|\xi_3 - \xi_1| \sim 1$, we have that $\phi \sim \sin\phi \sim 2^j\theta$. If we choose $t \in \mathbb{Z}$ so that $2^{-t} < 2^j\theta \leq 2^{-t+1}$, then $\theta \sim 2^{-(j+t)}$ and $\phi \sim 2^{-t}$. By the sine law, the acute angle between $l(\xi_1, \xi_2)$ and $l(\xi_2, \xi_3)$ is $\sim \phi$. We want to use these constraints to get simpler triplets of surfaces preserving transversality.

First, we will cover $U_1 \times U_2$ with sets of the form $(U_1 \cap L) \times (U_2 \cap L)$, where L are bands of width $\sim 2^{-(j+t)}$, which allows us to control the rotation of the line $l(\xi_1, \xi_2)$ when $\xi_1 \in U_1 \cap L$ and $\xi_2 \in U_2 \cap L$. We take for reference the band $L_{\text{ref}} = \mathbb{R} \times [0, \frac{1}{10}2^{-(j+t)}]$ and tile the whole plane with parallel translations $L_0 = \mathbb{R} \times [\frac{l}{10}2^{-(j+t)}, \frac{l+1}{10}2^{-(j+t)}]$ of L_{ref}, for $l \in \mathbb{Z}$. We rotate L_{ref} by a small angle, say $\omega = \frac{\pi}{50}2^{-t}$, and tile again the whole plane as before, with parallel translations of the rotated band $L_{\text{ref},\omega}$. We repeat this process with all the angles $\omega \in \frac{\pi}{50}2^{-t}\mathbb{Z} \cap [0, 2\pi]$, until we have covered $U_1 \times U_2$ with sets $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$. This is possible, because the distance between $\tau_1^{j,j}$ and $\tau_2^{j,j}$ is $\sim 2^{-j}$. Furthermore, given any four points $\xi_k, \xi_k' \in \tau_k^{j,j} \cap L_\omega$, the angle between $l(\xi_1, \xi_2)$ and $l(\xi_1', \xi_2')$ is at most $\frac{1}{10}2^{-t}$.

Now we will decompose U_3 into sets $U_3 \cap \Gamma$, so that $\tau_1^{j,j} \cap L_\omega$, $\tau_2^{j,j} \cap L_\omega$ and $U_3 \cap \Gamma$ are θ-transverse. To do that, fix a band L_ω and denote by $\bar{\xi}_1$ the midpoint of $\tau_1^{j,j} \cap l_\omega$, where l_ω is the central axis of L_ω. We cover U_3 with sets $\tau_3 \cap \Gamma_{\phi, L_\omega}$, where Γ_{ϕ, L_ω} is the set of points ξ_3 such that the acute angle between $l(\bar{\xi}_1, \xi_3)$ and l_ω lies in the interval $(\phi - \frac{\pi}{20}2^{-t}, \phi + \frac{\pi}{20}2^{-t})$. Notice that for $\phi \sim 2^{-t}$, $\xi_1 \in U_1$ and $\xi_3 \in U_3$, the line $l(\xi_1, \xi_3)$ cannot rotate more than $c2^{-t}$. Since for any pair of points $\xi_k \in \tau_k^{j,j} \cap L_\omega$ the line $l(\xi_1, \xi_2)$ cannot rotate more that $\frac{1}{10}2^{-t}$, the area of the triangle spanned by points $\xi_1 \in \tau_1^{j,j} \cap L_\omega$, $\xi_2 \in \tau_2^{j,j} \cap L_\omega$ and $\xi_3 \in \tau_3 \cap \Gamma_{\phi, L_\omega}$, for $\phi \sim 2^{-t}$, is $\sim 2^{-(j+t)}$. Hence, given L_ω we can cover $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \times U_3$ with $\lesssim 1$ disjoint sets $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \times (\tau_3 \cap \Gamma_{\phi, L_\omega})$, for $\phi \sim 2^{-t}$. Therefore, we get a covering

$$U_1 \times U_2 \times U_3 \subset \bigcup_{L_\omega, \phi \sim 2^{-t}} (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \times (\tau_3 \cap \Gamma_{\phi, L_\omega}).$$

It is important to keep the number of pieces in our decomposition small, so that we can control them in (2-3) with the triangle inequality.

The sets $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$ could be quite large in number, around 2^{2t}, however the number of these sets needed to cover $U_1 \times U_2$ is not that large. If $t \leq 2$ then $2^{2t} = O(1)$ and we are done, so we assume that $t > 2$ and take as reference three points $\xi_k \in U_k$. For every point $\xi_2' \in U_2$ the acute angle ϕ between $l(\xi_1, \xi_2')$ and $l(\xi_1, \xi_3)$ is necessarily $\sim 2^{-t}$, hence U_2 must be contained in band of width $\sim 2^{-(j+t)}$ with principal axis $l(\xi_1, \xi_2)$ (recall that U_1 and U_2 are 2^{-j}-separated). Likewise, repeating the previous reasoning but permuting 1 and 2, the set U_1 is contained in a band of width $\sim 2^{-(j+t)}$ with principal axis $l(\xi_1, \xi_2)$. Hence, the number of sets $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$ needed to cover $U_1 \times U_2$ is $\lesssim 1$. Let us denote this collection of bands by $\mathcal{L}$, so we get a covering

$$U_1 \times U_2 \times U_3 \subset \bigcup_{L_\omega \in \mathcal{L}, \phi \sim 2^{-t}} (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \times (\tau_3 \cap \Gamma_{\phi, L_\omega}),$$

where the number of sets is $\lesssim 1$.

Unfortunately, our collection of sets $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$, for $L_\omega \in \mathcal{L}$, is not pairwise disjoint, but overlap finitely, so we must go through a laborious technical process to amend this. We divide $\mathcal{L}$ into sets $\mathcal{L}_1, \ldots, \mathcal{L}_C$, such that the angle between every pair of bands in the same $\mathcal{L}_k$ is $> C2^{-t}$, for $C \gg 1$, so that the sets $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$, for $L_\omega \in \mathcal{L}_k$, are pairwise disjoint. Now, we can take the pairwise disjoint sets $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$, for $L_\omega \in \mathcal{L}_1$, as part of our covering and we write $\Omega_1 = \bigcup_{L_\omega \in \mathcal{L}_1} (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$. We go on covering the points in

$$\bigcup_{L_\omega \in \mathcal{L}_2} (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \setminus \Omega_1 = \bigcup_{L_\omega \in \mathcal{L}_2} \bigcap_{\substack{L_{\omega'} \in \mathcal{L}_1, \\ L_{\omega'} \cap L_\omega \neq \emptyset}} (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \setminus (\tau_1^{j,j} \cap L_{\omega'}) \times (\tau_2^{j,j} \cap L_{\omega'})$$

In general, given four sets A_k, we have the identities

$$(A_1 \times A_2) \cap (A_3 \times A_4) = (A_1 \cap A_3) \times (A_2 \cap A_4), \tag{2-6}$$

$$(A_1 \times A_2) \setminus (A_3 \times A_4) = (A_1 \setminus A_3 \times A_2 \cap A_4) \cup (A_1 \cap A_3 \times A_2 \setminus A_4) \cup (A_1 \setminus A_3 \times A_2 \setminus A_4), \tag{2-7}$$

where in (2-7), the sets at the right are pairwise disjoint. Hence, we can write $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \setminus (\tau_1^{j,j} \cap L_{\omega'}) \times (\tau_2^{j,j} \cap L_{\omega'})$ as a union of three pairwise disjoint sets $E_{i,L_\omega,L_{\omega'}} \times F_{i,L_\omega,L_{\omega'}}$ for $i = 1, 2, 3$, $L_\omega \in \mathcal{L}_2$ and $L_{\omega'} \in \mathcal{L}_1$. Hence, we have

$$\bigcap_{\substack{L_{\omega'} \in \mathcal{L}_1, \\ L_{\omega'} \cap L_\omega \neq \emptyset}} (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \setminus (\tau_1^{j,j} \cap L_{\omega'}) \times (\tau_2^{j,j} \cap L_{\omega'}) = \bigcap_{\substack{L_{\omega'} \in \mathcal{L}_1, \\ L_{\omega'} \cap L_\omega \neq \emptyset}} \bigcup_{i=1}^{3} E_{i,L_\omega,L_{\omega'}} \times F_{i,L_\omega,L_{\omega'}}.$$

By (2-6) and some additional set-theoretic considerations, the term at the right hand side is the disjoint union of sets of the form $G \times H \subset (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$, for $L_\omega \in \mathcal{L}_2$.[4] Hence, we can write $\Omega_2 = \bigcup_{L \in \mathcal{L}_1 \cup \mathcal{L}_2} (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$ as a pairwise disjoint union of sets contained in $(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega)$, for some $L_\omega \in \mathcal{L}_1 \cup \mathcal{L}_2$. We do the same with $\bigcup_{L_\omega \in \mathcal{L}_3} (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \setminus \Omega_2$ and continue inductively, until we cover $U_1 \times U_2 \times U_3$. Since the number of sets and steps involved in the process is small in number, the final number of sets is $\lesssim 1$.

[4] The exact form of $G \times H$ is irrelevant, but perhaps the reader wants to verify the details. For each map $s : \{L_{\omega'} \in \mathcal{L}_1 \mid L_{\omega'} \cap L_\omega \neq \emptyset\} \mapsto \{1, 2, 3\}$ define the set $G_s = \bigcap_{L_{\omega'}} E_{s(L_{\omega'}),L_\omega,L_{\omega'}}$, and likewise define H_s, with F instead of E. Then, the aforementioned union of $G_s \times H_s$ runs over all the functions s.

Definition 2.2. We say that a triplet $U_1 \times U_2 \times U_3$ is (j,t)-standard if it can be written as

$$(\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \times (\tau_3 \cap \Gamma_{\phi,L_\omega}), \tag{2-8}$$

where

- L_ω is a band of width $\frac{1}{10}2^{-(j+t)}$.
- Γ_{ϕ,L_ω} is a band of width $\frac{\pi}{20}2^{-t}$ making an angle $\phi \sim 2^{-t}$ with L_ω.
- $\tau_1^{j,j}$ and $\tau_2^{j,j}$ are close squares of side length 2^{-j}.
- τ_3 is a unit square at distance ~ 1 from $\tau_1^{j,j}$ and $\tau_2^{j,j}$.

Recall that the area spanned by points in a (j,t)-standard triplet is $\sim 2^{-(j+t)}$. Therefore, the Lemma 2.1 and the scaling at the end of the previous subsection, plus the fine decomposition in this subsection and the triangle inequality, allow us to conclude the following Lemma.

Lemma 2.3. *The multilinear inequality in Theorem 1.7 is a consequence of the trilinear inequality*

$$\int_{B_R} |\prod_{k=1}^{3} \mathcal{E} f_k|\, dx \le C(\log R)^C \theta^{-1/2} \prod_{k=1}^{3} \|f_k\|_2, \tag{2-9}$$

where the functions f_k are supported in a (j,t)-standard triplet for $\theta \sim 2^{-(j+t)}$.

2.2 Orthogonality

In the previous section, Lemma 2.3, we showed that it suffices to prove the trilinear bound for standard triplets $U_1 \times U_2 \times U_3 = (\tau_1^{j,j} \cap L_\omega) \times (\tau_2^{j,j} \cap L_\omega) \times (\tau_3 \cap \Gamma_{\phi,L_\omega})$; see Definition 2.2. Unlike the paraboloid, we do not have rotational symmetry and we must take care of the acute angle ω between L_ω and the coordinate axis ξ_1. By our construction $\frac{1}{\sqrt{17}} \le \sin\omega \le \frac{4}{\sqrt{17}}$. We rotate the plane so that L_ω is now parallel to the coordinate axis and modify the extension operator accordingly to obtain

$$\mathcal{E} f(x) = \int f(\xi) e(-\langle x', \xi\rangle - \frac{1}{2} x_3 (a(\xi_1^2 - \xi_2^2) + 2b\xi_1\xi_2))\, d\xi,$$

where $a = \sin(2\omega)$ and $b = \cos(2\omega)$. It will be important that $a \ge 8/17$, and this would not be guaranteed if we had not introduced the coarse decomposition of Subsection 2.1.1 that is adapted to the hyperbolic problem. By translation symmetry, we can assume that the origin is the midpoint of $\tau_1^{j,j} \cap l_\omega$, where l_ω is the central axis of L_ω.

Let us summarize some important properties of this rotated and translated standard triplet $U_1 \times U_2 \times U_3$ (the last property is only important for the Appendix):

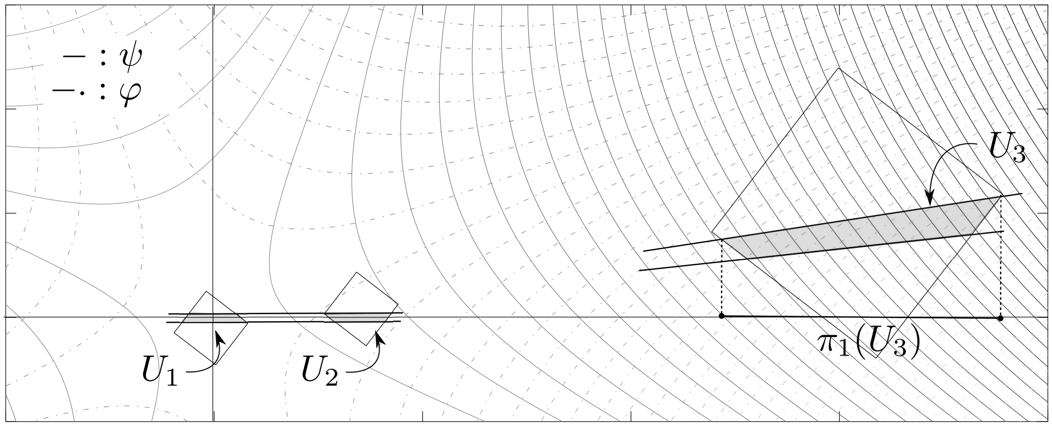

Figure **2-4**: The level curves of ψ and φ are depicted for reference. The sets U_1, U_2 and U_3 are a rotated and translated standard triplet. We see that U_1 and U_2 are separated from the projection $\pi_1(U_3)$. Here $a = \sin(3\pi/4)$ and $b = \cos(3\pi/4)$.

a) The area spanned by points in these sets is $\sim 2^{-(j+t)} \sim \theta$.

b) $U_1 \cup U_2 \subset B(0, C2^{-j}) \cap \{\xi = (\xi_1, \xi_2) \mid |\xi_2| \leq \frac{1}{10}2^{-(j+t)}\}$.

c) If $\xi_1 \in U_1$ and $\xi_2 \in U_2$, then $|\xi_1 - \xi_2| > (\sqrt{2})2^{-j}$.

d) If $\xi = (\xi_1, \xi_2) \in U_3$ then $|\xi| \sim 1$ and $\xi_2 = O(2^{-t})$.

e) If $2^{-t} \ll 1$, then the distance between the sets U_k, for $k = 1, 2$, and the projection $\pi_1(U_3)$ of U_3 into the first coordinate is ~ 1; see Figure **2-4**.

We pass now to the initial wave-packet decomposition. We will cover our sets U_k with thin caps of the following form:

Definition 2.4. A δ-cap α, for $0 < \delta \leq 1$, is the translation by a vector $c_\alpha \in \mathbb{R}^2$, called the center of the cap, of the set:

- If the cap is inside U_k for $k = 1, 2$:

$$\delta\mathcal{C}_k = \{l_1(a, b) + l_2(b, -a) \mid |l_1| \leq \frac{\delta}{10}2^{-(j+2t)} \text{ and } |l_2| \leq \frac{\delta}{10}2^{-(j+t)}\}. \tag{2-10}$$

- If the cap is inside U_3:

$$\delta\mathcal{C}_3 = \{l_1\nabla\varphi(c_\alpha) + l_2\nabla\psi(c_\alpha) \mid |l_1| \leq \frac{\delta}{10}2^{-2(j+t)} \text{ and } |l_2| \leq \frac{\delta}{10}2^{-(j+t)}\}, \tag{2-11}$$

where $\varphi(\xi) = a(\xi_1^2 - \xi_2^2) + 2b\xi_1\xi_2$ and $\psi(\xi) = b(\xi_1^2 - \xi_2^2) - 2a\xi_1\xi_2$.

Remark. The definition of $\mathcal{C}_3$ is more involved and we need to use the pair of functions φ (our model phase) and ψ. It is useful to note that φ and ψ are a pair of conjugate functions; in fact, if we write $z = \xi_1 + i\xi_2$ then $(e^{2\pi i\phi} z)^2 = (\psi(\xi), \varphi(\xi))$. Hence, we get a conformal change of coordinates $\eta_1 = \psi(\xi)$ and $\eta_2 = \varphi(\xi)$, whenever we are in a disk away from the origin. Hence, we see that $\nabla\varphi(\xi)$ and $\nabla\psi(\xi)$ are orthogonal and additionally that $|\nabla\varphi(\xi)| = |\nabla\psi(\xi)| = 2|\xi| \sim 1$ for $\xi \in U_3$, so we may write

$$\delta\mathcal{C}_3 + c_\alpha = \{\xi \mid \varphi(\xi) - \varphi(c_\alpha) = O(\delta 2^{-2(j+t)}) \text{ and } \psi(\xi) - \psi(c_\alpha) = O(\delta 2^{-(j+t)})\}.$$

As we announced at the end of the introduction, if we choose carefully the caps $\alpha_k \subset U_k$, then the functions $f_k = \mathbb{1}_{\alpha_k}$ attain the sharp constant in the trilinear inequality (1-31). In particular, the caps we just defined do it.[5] The functions $\mathcal{E}\mathbb{1}_{\alpha_k}$ are essentially supported in tubes with elliptical cross-section and length $\delta^{-2}2^{2(j+t)}$. Moreover, these tubes intersect essentially in the dual parallelepiped $\delta^{-1}P^*$, where

$$P = \{s_1(a,b,0) + s_2(b,-a,0) + s_3(0,0,1) \mid |s_1| \le 2^{-(j+2t)}, |s_2| \le 2^{-(j+t)}, |s_3| \le 2^{-2(j+t)}\}, \tag{2-12}$$

and

$$P^* = \{s_1(a,b,0) + s_2(b,-a,0) + s_3(0,0,1) \mid |s_1| \le 2^{j+2t}, |s_2| \le 2^{j+t}, |s_3| \le 2^{2(j+t)}\}. \tag{2-13}$$

The above can be seen using the uncertainty principle, combined with the following Lemma.

Lemma 2.5. *If $\alpha_k \subset U_k$ is a δ-cap as described in Definition 2.4, then $\tilde{\alpha}_k \subset \tilde{c}_\alpha + \delta P$ (recall that $\tilde{\alpha}$ refers to the lift of the cap to the hyperbolic paraboloid).*

Proof. Let us begin with a cap α centered inside U_1 or U_2. We must verify that $(\xi - c_\alpha, \frac{1}{2}\varphi(\xi) - \frac{1}{2}\varphi(c_\alpha)) \in \delta P$ whenever[6] $\xi \in c_\alpha + \delta\mathcal{C}_k$. It is obvious from (2-10) and (2-12) that the first two coordinates satisfy this requirement, thus we turn to the last coordinate $(0,0,1)$. By expanding φ as a Taylor series, we get

$$\varphi(\xi) = \varphi(c_\alpha) + 2c_{\alpha,1}l_1 + 2c_{\alpha,2}l_2 + O(l_1^2 + l_2^2) = \varphi(c_\alpha) + O(2^{-2(j+t)}).$$

In the last step, we used the property (b) at the beginning of this section, namely, $c_{\alpha,1} = O(2^{-j})$ and $c_{\alpha,2} = O(2^{-(j+t)})$. This concludes the verification for caps in U_1 or U_2, so it remains to deal with caps centered inside U_3.

We must verify that $(\xi - c_\alpha, \frac{1}{2}\varphi(\xi) - \frac{1}{2}\varphi(c_\alpha)) \in \delta P$ whenever $\xi \in c_\alpha + \delta\mathcal{C}_3$. Let us start by considering the condition $|\langle \xi - c_\alpha, (a,b)\rangle| \lesssim \delta 2^{-(j+2t)}$, or what amounts to the same (see the definition of $\mathcal{C}_3$ in (2-11))

$$|\langle l_1\nabla\varphi(c_\alpha) + l_2\nabla\psi(c_\alpha), (a,b)\rangle| = 2|l_1 c_{\alpha,1} + l_2 c_{\alpha,2}| \lesssim \delta 2^{-(j+2t)}.$$

[5] There are other caps doing it, but they do not fit well in our construction.

[6] It the following kind of arguments, usually one needs to consider a mild dilatation, *e.g.* δCP, of the parallelepipeds involved, but at the end this is unimportant and we can gloss over this technicality.

Here we used the property (d) at the beginning of this section, namely, $c_{\alpha,1} = O(1)$ and $c_{\alpha,2} = O(2^{-t})$. Now we have to verify $|\langle \xi - c_\alpha, (b,-a)\rangle| \lesssim \delta 2^{-(j+t)}$, or

$$|\langle l_1 \nabla\varphi(c_\alpha) + l_2 \nabla\psi(c_\alpha), (b,-a)\rangle| = 2|l_1 c_{\alpha,2} + l_2 c_{\alpha,1}| \lesssim \delta 2^{-(j+t)}.$$

Recall that $|\nabla\varphi(c_\alpha)| = 2|c_\alpha| \sim 1$, so for the last direction $(0,0,1)$ we have, after expanding φ in a Taylor series, that

$$\varphi(\xi) = \varphi(c_\alpha) + 4l_1|c_\alpha|^2 + O(l_1^2 + l_2^2) = \varphi(c_\alpha) + O(\delta 2^{-2(j+t)}).$$

□

This Lemma implies that if $\mathbb{1}_{\alpha_k}$ are the indicator functions of δ-caps in U_k, then by the uncertainty principle we have that $|\mathcal{E}\mathbb{1}_{\alpha_k}(x)| \sim |\alpha_k|$ if $x \in \delta^{-1}P^*$, where P^* is the dual parallelepiped of P; see also Knapp's example [76, p. 46–47]. Hence, $\int_{\delta^{-1}P^*} |\mathcal{E}\mathbb{1}_{\alpha_1}\mathcal{E}\mathbb{1}_{\alpha_2}\mathcal{E}\mathbb{1}_{\alpha_3}|\,dx \gtrsim |\delta^{-1}P^*|\prod_k |\alpha_k|$ and $\prod_k \|\mathbb{1}_{\alpha_k}\|_2 = \prod_k |\alpha_k|^{1/2}$, so for every $\delta > 0$

$$\sup_{f_k \in L^2(U_k)} \frac{\int |\mathcal{E}f_1\mathcal{E}f_2\mathcal{E}f_3|\,dx}{\|f_1\|_2\|f_2\|_2\|f_3\|_2} \geq \frac{\int_{\delta^{-1}P^*} |\mathcal{E}\mathbb{1}_{\alpha_1}\mathcal{E}\mathbb{1}_{\alpha_2}\mathcal{E}\mathbb{1}_{\alpha_3}|\,dx}{\prod_k \|\mathbb{1}_{\alpha_k}\|_2} \gtrsim |\delta^{-1}P^*|\prod_k |\alpha_k|^{1/2} \sim \theta^{-1/2}, \tag{2-14}$$

hence the sharp constant in (1-31) is at least a multiple of $\theta^{-1/2}$, and in Theorem 1.7 we claim that it is at most a multiple of $\theta^{-1/2}$, up to logarithmic losses. We have thus that the functions $\mathbb{1}_{\alpha_k}$ realize the sharp constant in (1-31), so a decomposition of $\mathcal{E}f_k$ into wave-packets of the type $\mathcal{E}(f_k\mathbb{1}_{\alpha_k})$ is a union of functions essentially supported in tubes with elliptical cross-section and length $\delta^{-2}2^{2(j+t)}$, such that three tubes corresponding to the decomposition of U_1, U_2 and U_3 intersect essentially in a parallelepiped $\delta^{-1}P^*$. With this at hand, the multiscale analysis sketched in the introduction is efficient when changing scales, as we will show in the next section, therefore the only lacking piece of the inductive process is the first step, that is, the trilinear estimate localized to P^*, and to complete it we need an argument of orthogonality.

Lemma 2.6. *If f_k are functions supported in (j,t)-standard triplets, then*

$$\|\mathcal{E}f_1\mathcal{E}f_2\mathcal{E}f_3\|_{L^1(P^*)} \leq C\theta^{-\frac{1}{2}}\|f_1\|_2\|f_2\|_2\|f_3\|_2, \tag{2-15}$$

where C does not depend on j, t or the triplets.

Proof. Recall that we work over a rotated and translated standard triplet, as explained at the beginning of this section. We cover U_1 and U_2 with 1-caps $\alpha_k = (w_k, 0) + \mathcal{C}_k$, where $\alpha_k \in U_k$ and $(w_k, 0) = c_{\alpha_k}$ is the center of the cap. Since $\frac{3}{5} \leq a \leq 1$, we have for two caps $\alpha_k, \alpha_k' \subset U_k$ that $|w_k - w_k'| \geq \frac{1}{10a}2^{-(j+2t)} \sim 2^{-(j+2t)}$. Let us write $\mathcal{E}f_\alpha = \mathcal{E}(f\mathbb{1}_\alpha)$. First, we want to prove that

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{L^2(P^*)} \lesssim \Big(\sum_{\alpha_1,\alpha_2} \|\mathcal{E}f_{1,\alpha_1}\mathcal{E}f_{2,\alpha_2}\|^2_{L^2(\zeta_{P^*})}\Big)^{\frac{1}{2}}. \tag{2-16}$$

This is a refinement of the classical argument of L^4 orthogonality. In fact,

$$\begin{aligned}\|\mathcal{E}f_1\mathcal{E}f_2\|_{L^2(P^*)} &\lesssim \|\sum_{\alpha_1}\mathcal{E}f_{1,\alpha_1}\sum_{\alpha_2}\mathcal{E}f_{2,\alpha_2}\zeta_{P^*}\|_2,\\ &= \|\sum_{\alpha_1,\alpha_2}(\mathcal{E}f_{1,\alpha_1})^\vee * (\mathcal{E}f_{2,\alpha_2})^\vee * \check{\zeta}_{P^*}\|_2.\end{aligned}$$

It is not hard to see that $(\mathcal{E}f_{k,\alpha_k})^\vee = f_{k,\alpha_k}d\sigma$, where $f_{k,\alpha_k}d\sigma$ refers to the lift of f_{k,α_k} to a measure over $\tilde{\alpha}_k \subset S$ (see in Notations). Hence, we have to show that the supports of the different $f_{1,\alpha_1}d\sigma * f_{2,\alpha_2}d\sigma * \check{\zeta}_{P^*}$ overlap finitely. By lemma 2.5, the support of $f_{k,\alpha_k}d\sigma$ lies inside $((w_k,0),\frac{1}{2}\varphi(w_k,0)) + P$, so we have that the support of $f_{1,\alpha_1}d\sigma * f_{2,\alpha_2}d\sigma * \check{\zeta}_{P^*}$ lies in the Minkowski sum

$$(w_1+w_2,0,\frac{1}{2}\varphi(w_1,0)+\frac{1}{2}\varphi(w_2,0))+3P,$$

where $\varphi(w,0)=aw^2$. Hence, we have to prove that if we are given four caps $\alpha_k,\alpha_k'\subset U_k$ with centers $(w_k,0)$ and $(w_k',0)$, then the condition

$$(w_1+w_2-w_1'-w_2',0,\frac{1}{2}\varphi(w_1,0)+\frac{1}{2}\varphi(w_2,0)-\frac{1}{2}\varphi(w_1',0)-\frac{1}{2}\varphi(w_2',0))\in 6P$$

implies that the caps α_k and α_k' are nearly the same. Otherwise stated, the conditions

$$a(w_1+w_2-w_1'-w_2') = O(2^{-(j+2t)}), \tag{2-17}$$
$$b(w_1+w_2-w_1'-w_2') = O(2^{-(j+t)}), \tag{2-18}$$
$$a(w_1^2+w_2^2-(w_1')^2-(w_2')^2) = O(2^{-2(j+t)}), \tag{2-19}$$

imply that $w_k - w_k' = O(2^{-(j+2t)})$. We note that the condition (2-18) is weaker than (2-17). We write (2-17) as $a(w_1-w_1') = -a(w_2-w_2')+O(2^{-(j+2t)})$ and replace it in (2-19), using that $|w_k|\lesssim 2^{-j}$, to get

$$a(w_2'-w_2)(w_1+w_1'-w_2-w_2') = O(2^{-2(j+t)});$$

by the separation between points in U_1 and U_2, see property (c) at the beginning of this section, we have that $|w_1+w_1'-w_2-w_2'|\gtrsim 2^{-j}$, so we get $w_2'-w_2 = O(2^{-(j+2t)})$. Similarly $w_1'-w_1 = O(2^{-(j+2t)})$, which concludes the proof of (2-16).

We also have to prove

$$\|\mathcal{E}f_3\|_{L^2(P^*)} \lesssim \big(\sum_\alpha \|\mathcal{E}f_{3,\alpha}\|^2_{L^2(\zeta_{P^*})}\big)^{\frac{1}{2}}; \tag{2-20}$$

therefore, for different centers c_α of caps lying in U_3, we must show that the sets $\tilde{c}_\alpha + 2P$ overlap finitely. Equivalently, we must show that the conditions

$$\langle c_\alpha - c_{\alpha'},(a,b)\rangle = O(2^{-(j+2t)}), \tag{2-21}$$
$$\langle c_\alpha - c_{\alpha'},(b,-a)\rangle = O(2^{-(j+t)}), \tag{2-22}$$
$$\varphi(c_\alpha)-\varphi(c_{\alpha'}) = O(2^{-2(j+t)}), \tag{2-23}$$

imply $\varphi(c_\alpha)-\varphi(c_{\alpha'})=O(2^{-2(j+t)})$ and $\psi(c_\alpha)-\psi(c_{\alpha'})=O(2^{-(j+t)})$ (see the Remarks after Definition 2.4). We see that the former implication is immediately fulfilled from the condition (2-23). From (2-21) and (2-22) we have $|c_\alpha - c_{\alpha'}|=O(2^{-(j+t)})$. Since $|\nabla\varphi(c_{\alpha'})|=|\nabla\psi(c_{\alpha'})|=2|c_{\alpha'}|\sim 1$, we get

$$\begin{aligned}|c_\alpha-c_{\alpha'}|^2 &\sim |\langle\nabla\psi(c_{\alpha'}),c_\alpha-c_{\alpha'}\rangle\nabla\psi(c_{\alpha'})+\langle\nabla\varphi(c_{\alpha'}),c_\alpha-c_{\alpha'}\rangle\nabla\varphi(c_{\alpha'})|^2\\ &=|\langle\nabla\psi(c_{\alpha'}),c_\alpha-c_{\alpha'}\rangle\nabla\psi(c_{\alpha'})|^2+O(2^{-2(j+t)});\end{aligned}$$

hence $\langle\nabla\psi(c_{\alpha'}),c_\alpha-c_{\alpha'}\rangle=O(2^{-(j+t)})$.

We conclude now the Lemma using Hölder, the orthogonality in (2-16) and (2-20), the trivial bound $|\mathcal{E}f|\le|\mathrm{supp}\,f|^{1/2}\|f\|_2$ and the last relation in (2-14), to obtain

$$\begin{aligned}\Big\|\prod_k\mathcal{E}f_k\Big\|_{L^1(P^*)} &\le \|\mathcal{E}f_1\mathcal{E}f_2\|_{L^2(P^*)}\|\mathcal{E}f_3\|_{L^2(P^*)}\\ &\lesssim \Big(\sum_{\alpha_1,\alpha_2}\|\mathcal{E}f_{1,\alpha_1}\mathcal{E}f_{2,\alpha_2}\|^2_{L^2(\zeta_{P^*})}\Big)^{\frac12}\Big(\sum_{\alpha_3}\|\mathcal{E}f_{3,\alpha_3}\|^2_{L^2(\zeta_{P^*})}\Big)^{\frac12}\\ &\lesssim |P^*|\prod_k|\alpha_k|^{1/2}\Big(\sum_{\alpha_1,\alpha_2}\|f_{1,\alpha_1}\|_2^2\|f_{2,\alpha_2}\|_2^2\Big)^{\frac12}\Big(\sum_{\alpha_3}\|f_{3,\alpha_3}\|_2^2\Big)^{\frac12}\\ &\lesssim \theta^{-1/2}\prod_k\|f_k\|_2.\end{aligned}$$

□

2.3 Induction on Scales

The method of induction on scales, introduced by Bourgain in [3], was the first improvement over L^2 estimates and it is by now classical, but for the sake of completeness we write down the details, because we have to modify the argument to improve the efficiency when changing scales. As before, we are restricted to functions f_k supported on standard triplets in Definition 2.2.

Fix a rotated and translated standard triplet, and let $K(\delta^{-1})$ be the best constant that satisfies

$$\int_{\delta^{-1}P^*}|\prod_{k=1}^3\mathcal{E}f_k|\,dx\le K(\delta^{-1})\prod_{k=1}^3\|f_k\|_2,\qquad\text{for }\delta\le 1,$$

where the functions f_k are supported on the standard triplet and P^* is the dual parallelepiped (2-13). First, we aim to prove the inductive bound

$$K(R)\le C_0K(R^{\frac12}),\tag{2-24}$$

where C_0 does not depend on R nor θ. From (2-15) we deduce the base case bound $K(2)\le C\theta^{-\frac12}$.

We divide each U_k into the δ-caps α_k described in Definition 2.4, and recall that $f_{k,\alpha_k} = f_k \mathbb{1}_{\alpha_k}$. By the uncertainty principle $|\mathcal{E} f_{k,\alpha_k}|$ is roughly constant on translations of $\delta^{-1}P^*$, and we expect something like

$$\mathcal{E} f_k(x) \approx \sum_{\alpha_k} \mathcal{E} f_{k,\alpha_k}(z) e(-\langle x'-z', c_{\alpha_k}\rangle - \frac{1}{2}(x_3-z_3)\varphi(c_{\alpha_k})), \qquad \text{for } x \in z+\delta^{-1}P^*. \quad (2\text{-}25)$$

One may exploit this heuristic by using the reproducing formula

$$\mathcal{E} f_{k,\alpha_k} = \frac{1}{|\alpha_k|}\mathcal{E} h_{k,\alpha_k} * \big(\frac{1}{|\delta^{-1}P^*|}\zeta_{\delta^{-1}P^*}\mathcal{E}\mathbb{1}_{\alpha_k}\big), \quad (2\text{-}26)$$

where h_{k,α_k} is a function essentially equal to f_{k,α_k}. To see this precisely, we take inverse Fourier transform at both sides of the reproducing formula (2-26) and notice that

$$\frac{1}{|\alpha_k|}\big(\frac{1}{|\delta^{-1}P^*|}\zeta_{\delta^{-1}P^*}\mathcal{E}\mathbb{1}_{\alpha_k}\big)^\vee = \frac{1}{|\alpha_k|}\big(\frac{1}{|\delta^{-1}P^*|}\check{\zeta}_{\delta^{-1}P^*}\big) * \tilde{\mathbb{1}}_{\alpha_k},$$

where $(\mathcal{E}\mathbb{1}_{\alpha_k})^\vee = \tilde{\mathbb{1}}_{\alpha_k}$ and $\tilde{\mathbb{1}}_{\alpha_k}$ is the lift of $\mathbb{1}_{\alpha_k}$ to a measure carried by $\tilde{\alpha}_k \subset S$. The function $\frac{1}{|\delta^{-1}P^*|}\check{\zeta}_{\delta^{-1}P^*}$ is essentially supported in δP, and furthermore we can choose the function $\zeta_{\delta^{-1}P^*}$ so that $\frac{1}{5}\mathbb{1}_{2\delta P} \leq \frac{1}{|\delta^{-1}P^*|}\check{\zeta}_{\delta^{-1}P^*} \leq 1$. Hence, by Lemma 2.5 we get

$$R(x) := \frac{1}{|\alpha_k|}\big(\frac{1}{|\delta^{-1}P^*|}\check{\zeta}_{\delta^{-1}P^*}\big) * \tilde{\mathbb{1}}_{\alpha_k}(x) \sim 1, \qquad \text{for } x \in \tilde{\alpha}_k.$$

Therefore, we can define $\tilde{h}_{k,\alpha_k} := \tilde{f}_{k,\alpha_k}/R$.

Using the reproducing formula (2-26), Fubini, and the changes of variables $x \mapsto x+z$ and $y_k \mapsto y_k+z$, we get

$$\begin{aligned}
\int_{z+\delta^{-1}P^*} |\prod_{k=1}^{3}\mathcal{E} f_k|\, dx &= \int_{z+\delta^{-1}P^*} |\sum_{\alpha_1,\alpha_2,\alpha_3}\prod_{k=1}^{3}\mathcal{E} f_{k,\alpha_k}|\, dx \\
&= \int_{z+\delta^{-1}P^*} |\int \sum_{\alpha_1,\alpha_2,\alpha_3}\prod_{k}\Big(\frac{1}{|\delta^{-1}P^*|}\zeta_{\delta^{-1}P^*}(x-y_k) \\
&\qquad \frac{1}{|\alpha_k|}\mathcal{E}\mathbb{1}_{\alpha_k}(x-y_k)\mathcal{E} h_{k,\alpha_k}(y_k)\Big)\, dy_1dy_2dy_3|\, dx \\
&\leq \iint_{\delta^{-1}P^*} |\prod_k \frac{1}{|\delta^{-1}P^*|}\zeta_{\delta^{-1}P^*}(x-y_k) \\
&\qquad \prod_k\sum_{\alpha_k}\frac{1}{|\alpha_k|}\mathcal{E}\mathbb{1}_{\alpha_k}(x-y_k)\mathcal{E} h_{k,\alpha_k}(z+y_k)|\, dx\, dy_1dy_2dy_3.
\end{aligned} \quad (2\text{-}27)$$

We pause for a moment to note that

$$\begin{aligned}
\mathbb{1}_{\delta^{-1}P^*}(x)\prod_k\frac{1}{|\delta^{-1}P^*|}|\zeta_{\delta^{-1}P^*}(x-y_k)| &\leq \mathbb{1}_{\delta^{-1}P^*}(x)\prod_k\frac{1}{|\delta^{-1}P^*|}\sup_{x'\in\delta^{-1}P^*}|\zeta_{\delta^{-1}P^*}(x'-y_k)| \\
&= \mathbb{1}_{\delta^{-1}P^*}(x)\prod_k \eta_{\delta^{-1}P^*}(y_k),
\end{aligned}$$

where $\eta_{\delta^{-1}P^*}(y) := \frac{1}{|\delta^{-1}P^*|}\sup_{x'\in\delta^{-1}P^*}|\zeta_{\delta^{-1}P^*}(x'-y)|$; it is not hard to see that $\int \eta_{\delta^{-1}P^*} \lesssim 1$. Therefore, we can replace (2-27) by

$$\int_{z+\delta^{-1}P^*} \Big|\prod_{k=1}^{3}\mathcal{E}f_k\Big|\,dx \le \int \prod_k \eta_{\delta^{-1}P^*}(y_k) \Big[\int_{\delta^{-1}P^*}\Big|\prod_k\sum_{\alpha_k}\frac{1}{|\alpha_k|}\mathcal{E}\mathbb{1}_{\alpha_k}(x-y_k)\mathcal{E}h_{k,\alpha_k}(z+y_k)\Big|\,dx\Big]\,dy_1dy_2dy_3. \quad (2\text{-}28)$$

The weight $\eta_{\delta^{-1}P^*}$ has essentially restricted the support to $\delta^{-1}P^*$.

Now, to simplify further, we define the functions

$$g_k(\xi) = \sum_{\alpha_k}\frac{1}{|\alpha_k|}e(\langle y_k',\xi\rangle + \frac{1}{2}y_{k,3}\varphi(\xi))\mathcal{E}h_{k,\alpha_k}(z+y_k)\mathbb{1}_{\alpha_k}(\xi),$$

where y_k acts as a fixed parameter (one may compare (2-25) with $\mathcal{E}g_k$), to rewrite (2-28) as

$$\int_{z+\delta^{-1}P^*}\Big|\prod_{k=1}^{3}\mathcal{E}f_k\Big|\,dx \le \int\prod_k\eta_{\delta^{-1}P^*}(y_k)\Big[\int_{\delta^{-1}P^*}\Big|\prod_k\mathcal{E}g_k\Big|\,dx\Big]\,dy_1dy_2dy_3.$$

We apply the trilinear estimate to g_k to obtain

$$\begin{aligned}\int_{z+\delta^{-1}P^*}\Big|\prod_{k=1}^{3}\mathcal{E}f_k\Big|\,dx &\le K(\delta^{-1})\int\prod_k\eta_{\delta^{-1}P^*}(y_k)\prod_k\|g_k\|_2\,dy_1dy_2dy_3\\ &= K(\delta^{-1})\int\prod_k\eta_{\delta^{-1}P^*}(y_k)\Big[\prod_k|\alpha_k|^{-1/2}\Big(\sum_{\alpha_k}|\mathcal{E}h_{k,\alpha_k}(z+y_k)|^2\Big)^{1/2}\Big]\,dy_1dy_2dy_3.\end{aligned}$$

Integrating both sides over $\frac{1}{|\delta^{-1}P^*|}\int_{\delta^{-2}P^*}dz$ we get

$$\int_{\delta^{-2}P^*}\Big|\prod_{k=1}^{3}\mathcal{E}f_k\Big|\,dx \lesssim K(\delta^{-1})\frac{1}{|\delta^{-1}P^*|\prod_k|\alpha_k|^{1/2}}\int\prod_k\eta_{\delta^{-1}P^*}(y_k)\Big[\int_{\delta^{-2}P^*}\prod_k\Big(\sum_{\alpha_k}|\mathcal{E}h_{k,\alpha_k}(z+y_k)|^2\Big)^{1/2}dz\Big]\,dy_1dy_2dy_3, \quad (2\text{-}29)$$

where $|\delta^{-1}P^*|\prod_k|\alpha_k|^{1/2}\sim\theta^{-1/2}$ by the last relation in (2-14). Since the Fourier transform converts translations into modulations, that is,

$$\mathcal{E}h_{k,\alpha_k}(z+y_k) = \mathcal{E}(e(-\langle y_k',\xi\rangle - \frac{1}{2}y_{k,3}\varphi(\xi))h_{k,\alpha_k})(z),$$

it suffices to establish the following upper bound of the square function inside brackets in (2-29):

$$\int_{\delta^{-2}P^*}\prod_k\Big(\sum_{\alpha_k}|\mathcal{E}h_{k,\alpha_k}(z)|^2\Big)^{1/2}dz \lesssim \theta^{-\frac{1}{2}}\prod_k\|h_k\|_{L^2}. \quad (2\text{-}30)$$

Modulations of h_k do not alter the value of $\|h_k\|_{L^2}$.

Since the Fourier transform of $|\mathcal{E}h_{k,\alpha_k}|^2$ is supported in a rectangle of dimensions $\delta 2^{-(j+t)} \times \delta 2^{-(j+t)} \times \delta^2 2^{-2(j+t)}$, we have the reproducing formula $|\mathcal{E}h_{k,\alpha_k}|^2 = \frac{1}{|T_{\alpha_k}|}|\mathcal{E}h_{k,\alpha_k}|^2 * \zeta_{T_{\alpha_k}}$, where $\zeta_{T_{\alpha_k}}$ is essentially supported on a tube T_{α_k} of dimensions $\delta^{-1}2^{j+t} \times \delta^{-1}2^{j+t} \times \delta^{-2}2^{2(j+t)}$ and $|T_{\alpha_k}| := \int \zeta_{T_{\alpha_k}} \sim \delta^{-4}2^{4(j+t)}$. We are not using tubes with elliptical cross-section now, because we want to use the multilinear Kakeya inequality (1-17) on this collection of tubes, but since they do not have diameter 1, we have to make a change of variables so that Theorem 1.5 holds for tubes of diameter ρ; hence, the inequality (1-17) turns into

$$\int \Big(\prod_{k=1}^{n} \sum_{\alpha_k} \mu_{\alpha_k} * \mathbb{1}_{T_{\alpha_k}}\Big)^{\frac{1}{n-1}} dx \lesssim \rho^n \theta^{-\frac{1}{n-1}} \prod_{k=1}^{n} \Big(\sum_{\alpha_k} \|\mu_{\alpha_k}\|\Big)^{\frac{1}{n-1}}, \tag{2-31}$$

where μ_{α_k} are positive measures.

We can dominate $\zeta_{T_{\alpha_k}}$ by a weighted sum of translations of $\mathbb{1}_{T_{\alpha_k}}$ to get

$$|\mathcal{E}h_{k,\alpha_k}|^2 = \frac{1}{|T_{\alpha_k}|}|\mathcal{E}h_{k,\alpha_k}|^2 * \zeta_{T_{\alpha_k}} \leq \sum_{\omega} \frac{a_\omega}{|T_{\alpha_k}|}|\mathcal{E}h_{k,\alpha_k}|^2 * \mathbb{1}_{T_{\alpha_k}+\omega},$$

where the coefficients a_ω decay strongly when $|\omega| \to \infty$ and the translations $T_{\alpha_k} + \omega$ tile the space. Since $|\mathcal{E}h_{k,\alpha_k}|^2 * \mathbb{1}_{T_{\alpha_k}+\omega} = |\mathcal{E}h_{k,\alpha_k}|^2(\cdot - \omega) * \mathbb{1}_{T_{\alpha_k}}$ and translations are exchanged by modulation in the extension operator (see the discussion after (2-29)), we may restrict ourselves to the single weight at $\omega = 0$. This is because in the square function in (2-30) we have

$$\begin{aligned}\int_{\delta^{-2}P^*} \prod_k \Big(\sum_{\alpha_k} |\mathcal{E}h_{k,\alpha_k}|^2\Big)^{1/2} dz &\leq \int_{\delta^{-2}P^*} \prod_k \Big(\sum_{\omega}\sum_{\alpha_k} \frac{a_\omega}{|T_{\alpha_k}|}|\mathcal{E}h_{k,\alpha_k}|^2 * \mathbb{1}_{T_{\alpha_k}+\omega}\Big)^{1/2} dz \\ &\leq \sum_{\omega_1,\omega_2,\omega_3} a_{\omega_1}^{1/2}a_{\omega_2}^{1/2}a_{\omega_3}^{1/2} \int_{\delta^{-2}P^*} \prod_k \Big(\sum_{\alpha_k} \frac{1}{|T_{\alpha_k}|}|\mathcal{E}h_{k,\alpha_k}(z-\omega_k)|^2 * \mathbb{1}_{T_{\alpha_k}}\Big)^{1/2} dz.\end{aligned}$$

Hence, we can replace $\zeta_{T_{\alpha_k}}$ by $\mathbb{1}_{T_{\alpha_k}}$ in the square function. Since

$$(|\mathcal{E}h_{k,\alpha_k}|^2 * \mathbb{1}_{T_{\alpha_k}})\mathbb{1}_{\delta^{-2}P^*} \leq (|\mathcal{E}h_{k,\alpha_k}|^2 \mathbb{1}_{B(2\delta^{-2}2^{2(j+t)})}) * \mathbb{1}_{T_{\alpha_k}},$$

we can apply the multilinear estimate (2-31) for $\mu_{\alpha_k} = \frac{1}{|T_{\alpha_k}|}|\mathcal{E}h_{k,\alpha_k}|^2 \mathbb{1}_{B(2\delta^{-2}2^{2(j+t)})}$ to get

$$\int_{\delta^{-2}P^*} \prod_k \Big(\sum_{\alpha_k} |\mathcal{E}h_{k,\alpha_k}(z)|^2\Big)^{1/2} dz \lesssim (\delta^{-1}2^{j+t})^{-3}\theta^{-\frac{1}{2}} \prod_k \Big(\sum_{\alpha_k} \|\mathcal{E}h_{k,\alpha_k}\|^2_{L^2(B(2\delta^{-2}2^{2(j+t)}))}\Big)^{1/2}.$$

By Plancherel's identity we have that $\|\mathcal{E}f\|_{L^2(B(R))} \lesssim R^{\frac{1}{2}}\|f\|_2$, see [76, Thm. 7.4], and so we reach our desired estimate of the square function (2-30).

Plugging (2-30) in (2-29) and using $\int \eta_{\delta^{-1}P^*} \lesssim 1$ we get

$$\int_{\delta^{-2}P^*} \Big|\prod_{k=1}^{3} \mathcal{E}f_k\Big| dx \lesssim K(\delta^{-1})\Big(\int \eta_{\delta^{-1}P^*}\Big)^3 \prod_k \|h_k\|_{L^2} \lesssim K(\delta^{-1}) \prod_k \|f_k\|_{L^2},$$

which is what we wanted.

Now that we have proved the inductive bound (2-24), we iterate it to reach

$$K(R) \leq C_0^N K(R^{1/2^N}).$$

If we choose $N = \lfloor \log_2 \log_2 R \rfloor + 1$, then by our base case bound $K(2) \lesssim \theta^{-1/2}$ we have

$$K(R) \leq C(\log R)^C K(2) \lesssim (\log R)^C \theta^{-1/2}.$$

To conclude the proof of the inequality (1-31), we simply observe that $B_R \subset RP^*$.

2.4 Orthogonality between furthest pairs

Keeping in mind a possible application to the linear estimate, we present here the orthogonality between furthest pairs. This section is an extension of section 2.2, and we use thus the same notation. For reference, we copy here the important properties underlined there at the beginning: if $U_1 \times U_2 \times U_3$ is a rotated and translated (j,t)-standard triplet, then

a) The area spanned by points in these sets is $\sim 2^{-(j+t)} \sim \theta$.

b) $U_1 \cup U_2 \subset B(0, C2^{-j}) \cap \{\xi = (\xi_1, \xi_2) \mid |\xi_2| \leq \frac{1}{10} 2^{-(j+t)}\}$.

c) If $\xi_1 \in U_1$ and $\xi_2 \in U_2$, then $|\xi_1 - \xi_2| > (\sqrt{2})2^{-j}$.

d) If $\xi = (\xi_1, \xi_2) \in U_3$ then $|\xi| \sim 1$ and $\xi_2 = O(2^{-t})$.

e) If $2^{-t} \ll 1$, then the distance between the sets U_k, for $k = 1, 2$, and the projection $\pi_1(U_3)$ of U_3 into the first coordinate is ~ 1; see Figure **2-4**.

Proof of property (e). The distance between any pair of points $\xi_k \in U_k$ and $\xi_3 \in U_3$, for $k = 1, 2$, is ≥ 1. By construction, if $\xi_1 = (w, 0) \in U_1$ then the angle ϕ between $l(\xi_1, \xi_3)$ and the first coordinate axis is $\sim 2^{-t}$; this together with $|\xi_{3,2}|^2 + |\xi_{3,1} - w|^2 = |\xi_{3,1} - w|^2(\tan^2\phi + 1) \geq 1$ yields $|\xi_{3,1} - w| > \frac{1}{2}$ if $2^{-t} < \frac{1}{8}$ or $t > 3$. For $\xi_2 = (w', 0) \in U_2$ the acute angle between $l(\xi_2, \xi_3)$ and the first coordinate axis is necessarily $\sim 2^{-t}$, so the same argument allows us to conclude the proof of the property. □

We will prove orthogonality between U_1 and U_3; the case U_2 and U_3 is similar. We have to verify that for pairs of functions $\tilde{f}_{1,\alpha_1}$ and $\tilde{f}_{3,\alpha_3}$ over U_1 and U_3 respectively, the supports of the different $\tilde{f}_{1,\alpha_1} * \tilde{f}_{3,\alpha_3} * \zeta_P$ overlap finitely. By the proof of Lemma 2.5, to prove orthogonality we have to show that given four caps $\alpha_k, \alpha'_k \in U_k$, for $k = 1, 3$, the condition $\tilde{c}_{\alpha_1} + \tilde{c}_{\alpha_3} - \tilde{c}_{\alpha'_1} - \tilde{c}_{\alpha'_3} \in P$ implies that the caps α_k and α'_k are nearly the same. Otherwise stated, setting $(w, 0) = c_{\alpha_1}$ and $(w', 0) = c_{\alpha'_1}$, the conditions

$$a(w - w' + c_{\alpha_3,1} - c_{\alpha'_3,1}) + b(c_{\alpha_3,2} - c_{\alpha'_3,2}) = O(2^{-(j+2t)}), \tag{2-32}$$

$$b(w - w' + c_{\alpha_3,1} - c_{\alpha'_3,1}) - a(c_{\alpha_3,2} - c_{\alpha'_3,2}) = O(2^{-(j+t)}), \tag{2-33}$$

$$a(w^2 - w'^2) + \varphi(c_{\alpha_3}) - \varphi(c_{\alpha'_3}) = O(2^{-2(j+t)}). \tag{2-34}$$

imply that

$$w - w' = O(2^{-(j+2t)}), \tag{2-35}$$
$$\varphi(c_{\alpha_3}) - \varphi(c_{\alpha_3'}) = O(2^{-2(j+t)}), \tag{2-36}$$
$$\psi(c_{\alpha_3}) - \psi(c_{\alpha_3'}) = O(2^{-(j+t)}), \tag{2-37}$$

where $\varphi(\xi) = a(\xi_1^2 - \xi_2^2) + 2b\xi_1\xi_2$ and $\psi(\xi) = b(\xi_1^2 - \xi_2^2) - 2a\xi_1\xi_2$ are the pair of conjugate functions.

When $2^{-t} \sim 1$, *i.e.* $t \lesssim 1$, the orthogonality is a repetition of the arguments leading to (2-20). In fact, since $2^{-t} \sim 1$ and by property (b), the conditions (2-32) – (2-34) reduce to

$$a(c_{\alpha_3,1} - c_{\alpha_3',1}) + b(c_{\alpha_3,2} - c_{\alpha_3',2}) = O(2^{-j}),$$
$$b(c_{\alpha_3,1} - c_{\alpha_3',1}) - a(c_{\alpha_3,2} - c_{\alpha_3',2}) = O(2^{-j}),$$
$$\varphi(c_{\alpha_3}) - \varphi(c_{\alpha_3'}) = O(2^{-2j}).$$

We already showed in Section 2.2 that they imply the conclusions (2-35) – (2-37).

Therefore, we assume that $2^{-t} \ll 1$, so that the property (e) holds. Our first aim is to see that the centers c_{α_3} and $c_{\alpha_3'}$ are not far away from each other. After we are sure they are somewhat close, we can safely refine the argument to show that they satisfy our desired conclusions. We recommend to use Figure **2-4** as a geometrical aid in the steps to come.

We show first that $|c_{\alpha_3} - c_{\alpha_3'}| = O(2^{-(j+t)})$. From (2-32) and (2-33) we get $c_{\alpha_3,2} - c_{\alpha_3',2} = O(2^{-(j+t)})$. Next we choose $(\eta_1, \eta_2) \in \mathbb{R}^2$ such that

$$\eta_2 = c_{\alpha_3,2} \qquad \text{and} \qquad \varphi(\eta) = \varphi(c_{\alpha_3'}). \tag{2-38}$$

It is geometrically evident that $|\eta - c_{\alpha_3'}| = O(2^{-(j+t)})$, however let us prove it rigorously. Join η to $c_{\alpha_3'}$ with a path $r \mapsto (\xi_1(r), r)$ such that $\varphi(\xi_1(r), r) = \varphi(\eta)$. We differentiate to get $\frac{d}{dr}\xi_1(r) = -(\partial_2\varphi/\partial_1\varphi)(\xi_1(r), r)$, but $|\partial_2\varphi/\partial_1\varphi| = |b - a\frac{r}{\xi_1(r)}|/|a + b\frac{r}{\xi_1(r)}| \le 3$, where we used that $|\frac{r}{\xi_1(r)}| \lesssim 2^{-t} \ll 1$. By the mean value theorem we get

$$|\eta_1 - c_{\alpha_3',1}| = |\xi_1(c_{\alpha_3,2}) - \xi_1(c_{\alpha_3',2})| \lesssim |c_{\alpha_3,2} - c_{\alpha_3',2}| \lesssim O(2^{-(j+t)}).$$

From this, (2-38) and $c_{\alpha_3,2} - c_{\alpha_3',2} = O(2^{-(j+t)})$, we have that $|\eta - c_{\alpha_3'}| = O(2^{-(j+t)})$; we must still prove $|c_{\alpha_3} - \eta| = O(2^{-(j+t)})$.

We write (2-32) as $a(w - w' + c_{\alpha_3,1} - \eta_1) + a(\eta_1 - c_{\alpha_3',1}) + b(\eta_2 - c_{\alpha_3',1}) = O(2^{-(j+2t)})$ or, taking into account that $|\eta - c_{\alpha_3'}| = O(2^{-(j+t)})$, as

$$a(w - w' + c_{\alpha_3,1} - \eta_1) = O(2^{-(j+t)}).$$

We replace it in (2-34) to get, for some $\rho \in (c_{\alpha_3,1}, \eta_1)$, that

$$-a(w + w')(c_{\alpha_3,1} - \eta_1) + \varphi(c_{\alpha_3}) - \varphi(\eta) = O(2^{-(j+t)}), \text{ or}$$
$$(c_{\alpha_3,1} - \eta_1)[-a(w + w') + \partial_1\varphi(\rho, c_{\alpha_3,2})] = O(2^{-(j+t)}).$$

We observe that $|-a(w+w')+\partial_1\varphi(\rho,c_{\alpha_3,2})| = 2a|(w+w')/2-\rho(1+\frac{bc_{\alpha_3,2}}{a\rho})| \gtrsim 1$, where we used properties (d) and (e). Hence $c_{\alpha_3,1}-\eta_1 = O(2^{-(j+t)})$ and we use this together with $\eta_2 = c_{\alpha_3,2}$ to conclude that $|c_{\alpha_3}-c_{\alpha_3'}| = O(2^{-(j+t)})$, which was our first aim.

Now that we have finished the preparatory result, we are ready for our main aims (2-35) – (2-37). We will not repeat the derivation of (2-37), which was already done when we proved (2-20), thus we have $\langle\nabla\psi(c_{\alpha_3'}), c_{\alpha_3}-c_{\alpha_3'}\rangle = O(2^{-(j+t)})$ and we are left with the verification of $\varphi(c_{\alpha_3})-\varphi(c_{\alpha_3'}) = O(2^{-2(j+t)})$ and $w-w' = O(2^{-(j+2t)})$.

In what follows, it is useful to keep in mind that we are working in a small region of diameter $\lesssim 2^{-(j+t)}$, where α_3 and α_3' live in. We join c_{α_3} and $c_{\alpha_3'}$ with a piecewise differentiable path $\gamma:[0,s_2]\to\mathbb{R}^2$ such that

$$\gamma'(s) = \begin{cases} \frac{\nabla\varphi(\gamma(s))}{|\nabla\varphi(\gamma(s))|}, & \text{for } I_1 = [0,s_1), \\ \frac{\nabla\psi(\gamma(s))}{|\nabla\psi(\gamma(s))|}, & \text{for } I_2 = (s_1,s_2], \end{cases}$$

where $\varphi(\gamma(s_1)) = \varphi(c_{\alpha_3'})$ and $\psi(\gamma(s_2)) = \psi(c_{\alpha_3'})$. The path γ travels parallel to the axes defined by the coordinates $(\eta_1,\eta_2) = (\psi(\xi),\varphi(\xi))$, as we remarked after Definition 2.4.

We observe that $\varphi(c_{\alpha_3'})-\varphi(c_{\alpha_3}) = \int_{I_1}|\nabla\varphi(\gamma(s))|\,ds$, and because $|\nabla\varphi(\gamma(s))| = 2|\gamma(s)| \sim 1$, we have $|I_1|\sim|\varphi(c_{\alpha_3})-\varphi(c_{\alpha_3'})|$; likewise $|I_2|\sim|\psi(c_{\alpha_3})-\psi(c_{\alpha_3'})|$. If we write $c_{\alpha_3}-c_{\alpha_3'} = -\int_0^{s_2}\gamma'(s)\,ds$, then the conditions (2-32) – (2-34) take the form

$$a(w-w') - \int_{I_1}\frac{\gamma_1(s)}{|\gamma(s)|}\,ds + \int_{I_2}\frac{\gamma_2(s)}{|\gamma(s)|}\,ds = O(2^{-(j+2t)}), \tag{2-39}$$

$$b(w-w') - \int_{I_1}\frac{\gamma_2(s)}{|\gamma_2(s)|}\,ds - \int_{I_2}\frac{\gamma_1(s)}{|\gamma_1(s)|}\,ds = O(2^{-(j+t)}), \tag{2-40}$$

$$a(w^2-w'^2) - \int_{I_1}|\nabla\varphi(\gamma(s))|\,ds = O(2^{-2(j+t)}). \tag{2-41}$$

Since we already know that $|I_2|\sim|\psi(c_{\alpha_3})-\psi(c_{\alpha_3'})| = O(2^{-(j+t)})$, and by property (d) we have $\gamma_2(s)/|\gamma(s)| = O(2^{-t})$, the condition (2-39) turns into

$$a(w-w') - \int_{I_1}\frac{\gamma_1(s)}{|\gamma(s)|}\,ds = O(2^{-(j+2t)}). \tag{2-42}$$

Replacing in (2-41) we get

$$\int_{I_1}\frac{\gamma_1(s)}{|\gamma(s)|}\,ds(w+w') - \int_{I_1}|\nabla\varphi(\gamma(s))|\,ds = O(2^{-2(j+t)}), \text{ or}$$

$$2|I_1|\Big[\frac{1}{|I_1|}\int_{I_1}\frac{\gamma_1(s)}{|\gamma(s)|}\,ds\frac{w+w'}{2} - \frac{1}{|I_1|}\int_{I_1}|\gamma(s)|\,ds\Big] = O(2^{-2(j+t)}).$$

Since $\frac{1}{|I_1|}\int_{I_1}\frac{\gamma_1(s)}{|\gamma(s)|}\,ds\sim 1$ and $|\gamma(s)|\in U_3$, the property of separation (e) allows us to conclude that the term in parentheses is $\gtrsim 1$ and $|I_1| = O(2^{-2(j+t)})$, which is what we wanted. Finally, $w-w' = O(2^{-(j+2t)})$ follows from (2-42).

For more information about the utility of this kind of orthogonality, see the work of Ramos [60].

3 Decay of Spherical Means of the Fourier Transform of Measures

Given a set of N points in the plane, at least how many distinct distances are always determined by these points? It is very hard to give an exact answer for each N, however Erdős [26], after considering the example of a square grid $[0,\sqrt{N}]^2 \cap \mathbb{Z}^2$, conjectured that $N^{1-o(1)}$ should be a lower bound for the number of distinct distances. As regards an upper bound, it is not hard to see that a generic arrangement of N points determines $\frac{N(N-1)}{2} \sim N^2$ distinct distances. The number of distinct distances of a set E is denoted by

$$\Delta(E) := \{|x-y| \mid x, y \in E\},$$

so the problem is to prove that $|\Delta(E)| \geq CN^{1-o(1)}$ for some constant C independent of N. The conjecture remained open for almost 70 years, until it was finally solved by Guth and Katz [38], who showed that $|\Delta(E)| \geq CN/\log N$ by using the polynomial method.

If instead of a finite set of points, we consider a set E with positive Lebesgue measure, then a classical theorem of Steinhaus states that $|\Delta(E)| > 0$; here $|\cdot|$ denotes the Lebesgue measure of the set. Moreover, $\Delta(E) \supset [0,\delta)$ for some small enough $\delta > 0$. This can be proven by noting that the support of $\mathbb{1}_E * \mathbb{1}_{-E}$ contains all the vectors $x-y$ for $x, y \in E$, hence by $\mathbb{1}_E * \mathbb{1}_{-E}(0) = |E| > 0$ and by the continuity of the convolution of simple functions, we conclude that there is a ball B_δ inside of the support of $\mathbb{1}_E * \mathbb{1}_{-E}$.

Steinhaus' theorem says that the distance set of a set with positive Lebesgue measure is large, hence it is natural to ask at least how big should be a set, so that its distance set has positive Lebesgue measure. The continuous analogue of Erdős conjecture was explicitly posed by Falconer [27], who measured the size of a set by means of the Hausdorff dimension. To define the s-dimensional content $\mathcal{H}^s$ of a set $E \in \mathbb{R}^n$, we define first

$$\mathcal{H}^s_\delta(E) := \inf\{\sum_i r_i^s \mid E \subset \bigcup_i B_{r_i} \text{ and } r_i < \delta\}.$$

Since $\delta \mapsto \mathcal{H}^s_\delta$ is a non-increasing function, we can take the limit $\lim_{\delta\to 0^+} \mathcal{H}^s_\delta$ as the definition of the s-dimensional content $\mathcal{H}^s(E)$. For $s_1 < s_2$ and $r_i < \delta$ we have $\sum_i r_i^{s_2} \leq \delta^{s_2-s_1} \sum_i r_i^{s_1}$, therefore $\mathcal{H}^{s_2}_\delta < \delta^{s_2-s_1}\mathcal{H}^{s_1}_\delta$. If $\mathcal{H}^{s_1} < \infty$ then taking limit as $\delta \to 0^+$ we get that $\mathcal{H}^{s_2} = 0$. On the other hand, if $\mathcal{H}^{s_2} > 0$ then taking limit as $\delta \to 0^+$ we get that $\mathcal{H}^{s_1} = \infty$. Hence, we can define the (spherical) Hausdorff dimension of a set as

$$\dim(E) := \inf\{s \mid \mathcal{H}^s(E) = 0\},$$

or alternatively as

$$\dim(E) := \sup\{s \mid \mathcal{H}^s(E) = \infty\}.$$

For further information and properties of the Hausdorff dimension, see [29, 28, 52].

Falconer conjectured that if E has dimension $> n/2$, for $n \geq 2$, then $|\Delta(E)| > 0$. The conjecture is meaningless in $\mathbb{R}$, since there exist one dimensional sets such that $|\Delta(E)| = 0$. To show that $n/2$ is the critical dimension, Falconer used a fractal well known in number theory and similar in structure to the grid $[0, \sqrt{N}]^2 \cap \mathbb{Z}^2$. To define this set, let $q_1, q_2, \ldots$ be a sequence of integers increasing rapidly, say $q_{k+1} > q_k^k$, and define the sets $E_s = \bigcap_{k=1}^{\infty} E_{s,k} \subset \mathbb{R}$ for $0 < s < 1$, where

$$E_{s,k} := \{x \in [0,1] \mid |x - \frac{p}{q_k}| \leq q_k^{-1/s} \text{ for some } p \in \mathbb{Z}_+\}. \tag{3-1}$$

It is known that $\dim(E_s) = s$; see for example [28, sec. 8.5]. Taking cartesian products, we get sets $E = E_{s_1} \times \cdots \times E_{s_n} \subset \mathbb{R}^n$ of dimension $\geq s = s_1 + \cdots + s_n < n$. Let us assume that $s_i = s/n$ and that the sequence $\{q_k\}$ is the same for each axis, in which case $\dim(E) = s$. Since $\Delta(E) \subset \bigcap_k \Delta(E_k)$, where $E_k = E_{s/n,k} \times \cdots \times E_{s/n,k}$, we can focus on $\Delta(E_k)$. We want to prove that $s < n/2$ implies $|\Delta(E_k)| = 0$. In fact, since

$$\Delta(E_k) \subset \bigcup_R (\frac{R}{q_k} + B_{\sqrt{n} q_k^{-\frac{n}{s}}}),$$

where $R^2 = p_1^2 + \cdots + p_n^2 \in \mathbb{Z} \cap [0, n q_k^2]$, we have that $|\Delta(E_k)| \leq C_n q_k^{2-\frac{n}{s}}$. When $s < n/2$ the sequence $|\Delta(E_k)|$ tends to 0 as $k \to \infty$, hence $|\Delta(E)| = 0$ for $s < n/2$.

In [44], Kaufman introduced potential methods in the study of measure geometric properties of sets, and they have proven to be quite fruitful ever since. The potential of a positive measure μ is defined by

$$I_t(\mu) = \int |x - y|^{-t} \, d\mu(x) d\mu(y),$$

If we denote by $\mathcal{M}(E)$ the collection of Radon measures compactly supported in E such that $0 < \mu(E) < \infty$, then it is possible to give an equivalent definition of dimension for Borel sets, namely,

$$\dim(E) := \sup\{t \mid \text{ there exists } \mu \in \mathcal{M}(E) \text{ with } I_t(\mu) < \infty\}; \tag{3-2}$$

see [52, ch. 8]. As an example of these methods, we will prove the following theorem due to Falconer.

Theorem 3.1. *If E is a Borel set with dim$(E) > \frac{n+1}{2}$, then $|\Delta(E)| > 0$.*

Proof. Since $\dim(E) > \frac{n+1}{2}$, there exists a measure μ compactly supported in E such that $I_t(\mu) < \infty$ for $t > \frac{n+1}{2}$. We can use μ to measure the size of $\Delta(E)$. The "size" of the set of distances lying in the set $I \subset \mathbb{R}$ is $\delta(\mu)(I) := \int \mathbb{1}_I(|x - y|) \, d\mu(x) d\mu(y)$ and this, indeed, defines a measure compactly supported inside $\Delta(E)$, say $\operatorname{supp} \delta(\mu) \subset [0, 1]$.

Formally, for a measurable function g, we can write the chain of identities

$$\int \mu * g(y)\, d\mu(y) = \int (\mu * g)^\wedge(\xi)\bar{\hat{\mu}}(\xi)\, d\xi = \int \hat{g}(\xi)|\hat{\mu}(\xi)|^2\, d\xi; \tag{3-3}$$

this can be proven in the cases we need by a standard regularization procedure. We apply this identity to $g = \mathbb{1}_I$, for $I = (r, r+\epsilon)$, to get

$$\delta(\mu)(I) := \int \hat{\mathbb{1}}_I(\xi)|\hat{\mu}(\xi)|^2\, d\xi. \tag{3-4}$$

To estimate $\hat{\mathbb{1}}_I$ we can use our asymptotic expansion $\widehat{d\sigma}(\xi) \sim |\xi|^{-\frac{n-1}{2}}$, where $d\sigma$ is the standard measure over S^{n-1}, to get

$$\hat{\mathbb{1}}_I(\xi) = \int_{r<|x|<r+\epsilon} e(-\langle x, \xi\rangle)\, dx = \int_r^{r+\epsilon} \rho^{n-1}\widehat{d\sigma}(\rho\xi) d\rho \lesssim \int_r^{r+\epsilon} \frac{\rho^{n-1}}{|\rho\xi|^{\frac{n-1}{2}}} d\rho \lesssim \epsilon \frac{1}{|\xi|^{\frac{n-1}{2}}}.$$

We replace this estimate in (3-4) to get

$$\delta(\mu)(I) \lesssim \epsilon \int \frac{|\hat{\mu}(\xi)|^2}{|\xi|^{\frac{n-1}{2}}}\, d\xi. \tag{3-5}$$

Apply (3-3) to $g(x) = |x|^{-\beta}$ to reach

$$I_\beta(\mu) = c_{n,\beta} \int |\xi|^{-(n-\beta)}|\hat{\mu}(\xi)|^2\, d\xi; \tag{3-6}$$

here we used the identity $\widehat{K}_\alpha(\xi) = c_{\alpha,n}|\xi|^{-\alpha}$, where K_α is the Riesz potential $K_\alpha(x) = |x|^{-(n-\alpha)}$. Hence (3-5) and (3-6), for $\beta = \frac{n+1}{2}$, imply $\delta(\mu)(I) \le C_n I_{\frac{n+1}{2}}(\mu)\epsilon$.

Since $t > \frac{n+1}{2}$ we have $I_{\frac{n+1}{2}}(\mu) \le C I_t(\mu) < \infty$, hence $\delta(\mu)(I) \le C|I|$ for every $I \subset \mathbb{R}$. We conclude thus that $\delta(\mu)$ is absolutely continuous with respect to the Lebesgue measure and $\|D\delta(\mu)\|_\infty \lesssim 1$, where $D\delta(\mu)$ is the derivative of the measure $\delta(\mu)$, so $\Delta(E)$ must have positive Lebesgue measure. □

It is already clear in the theorem the relationship between Falconer's conjecture and the theory of restriction, through the asymptotic expansion for $\widehat{d\sigma}$. In the proof of Theorem 3.1 we showed that $I_s(\mu) < \infty$ for $s \ge \frac{n+1}{2}$ implies $D\delta(\mu) \in L^\infty$, but can we expect that $D\delta(\mu) \in L^\infty$ if $\frac{n}{2} < s < \frac{n+1}{2}$? In [51], Mattila answered this question in the negative, but this did not discourage him and tried to prove that $D\delta(\mu) \in L^2$ if $\frac{n}{2} < s < \frac{n+1}{2}$, which would suffice to show that $|\Delta(E)| > 0$; additional information can be consulted in [53]. It is worth mentioning that this is roughly the same strategy used by Guth and Katz in [38] when they count quadruples; see the next section.

The estimations of Mattila involve in a crucial way the spherical average decay of the Fourier transform of compactly supported measures, to be precise, he needed to consider the quantity

$$\sigma(\mu)(r) := \int_{S^{n-1}} |\hat{\mu}(r\omega)|^2\, d\sigma(\omega).$$

He proved the next theorem relating spherical decay and distance sets; see [54, prop. 15.3].

Theorem 3.2. *Let $E \subset \mathbb{R}^n$ be a Borel subset with $dim(E) > s$ and μ a finite measure compactly supported in E such that $I_s(\mu) < \infty$, and with $\sigma(\mu)(r) \leq Cr^{-t}$ for all $r > 0$. If $t + s \geq n$ then $|\Delta(E)| > 0$.*

Therefore, it is natural to define

$$\alpha(s) := \sup\{\alpha \mid \sigma(\mu)(r) \leq Cr^{-\alpha} \text{ for } r > 0, \operatorname{supp}\mu \subset B(0,1) \text{ and } I_s(\mu) < \infty\}.$$

A first estimate for $\alpha(s)$ is at hand: from (3-6) we get

$$I_s(\mu) = c\int_0^\infty r^{s-1}\sigma(\mu)(r)\, dr. \tag{3-7}$$

We choose μ such that $I_s(\mu) < \infty$ but $I_{s+\epsilon}(\mu) = \infty$ for some $\epsilon > 0$, thus it is impossible to get $\sigma(\mu)(r) \leq Cr^{-s-2\epsilon}$, otherwise $I_{s+\epsilon}(\mu)$ would be finite. We conclude that $\alpha(s) \leq s$ for every $0 < s < n$. It is neither very hard to show that $\alpha(s) \geq s$ for $0 < s \leq (n-1)/2$, and that $\alpha(s) \geq \frac{n-1}{2}$ for $(n-1)/2 \leq s < n$.

A better lower bound for $\alpha(s)$ is possible in the range $\frac{n+1}{2} < s < n$. For any Borel set E of dimension s there is a compactly supported measure μ such that $I_t(\mu) < \infty$ for $t < s$. To simplify, assume that $\operatorname{supp}\mu \subset B_1$. To estimate $\sigma(\mu)(r)$ we can use duality

$$\sigma(\mu)(r) = \frac{c_n}{r^{n-1}}\int |\hat{\mu}(\omega)|^2\, d\sigma_r(\omega) = \frac{c_n}{r^{n-1}}\sup_{\|g\|_2=1} |\int \hat{\mu}(\omega)g(\omega)\, d\sigma_r(\omega)|^2, \tag{3-8}$$

where $d\sigma_r$ is the standard measure over the sphere rS^{n-1}. Now it suffices, for $\|g\|_2 \leq 1$, to control

$$\int \hat{\mu}(\omega)g(\omega)\, d\sigma_r(\omega) = \int (gd\sigma_r)^\wedge d\mu.$$

We are dealing now with the extension operator $(gd\sigma_r)^\wedge$ associated to rS^{n-1}. By the reproducing formula $(gd\sigma_r)^\wedge = (gd\sigma_r)^\wedge * \hat{\zeta}_{B_r}$ and Fubini we have

$$|\int \hat{\mu}(\omega)g(\omega)\, d\sigma_r(\omega)| = |\int (gd\sigma_r)^\wedge \hat{\zeta}_{B_r} * \mu| \leq \|\hat{\zeta}_{B_r} * \mu\|_2 \|(gd\sigma_r)^\wedge\|_{L^2(B_1)}.$$

By Plancherel and (3-6) we have $\|\hat{\zeta}_{B_r} * \mu\|_2 = \|\zeta_{B_r}\hat{\mu}\|_2 \leq Cr^{\frac{n-t}{2}}\sqrt{I_t(\mu)}$. Since $(gd\sigma_r)^\wedge$ is the extension operator associated to rS^{n-1}, we can modify Lemma 1.1 to get $\|(gd\sigma_r)^\wedge\|_{L^2(B_1)} \leq C\|g\|_2 \leq C$, hence by replacing in the previous inequality we get

$$|\int \hat{\mu}(\omega)g(\omega)\, d\sigma_r(\omega)| \leq C_n r^{-\frac{t-n}{2}}\sqrt{I_t(\mu)}.$$

We replace it in (3-8) to conclude $\sigma(\mu)(r) \leq Cr^{-(t-1)}$ for $t < s$, therefore $\alpha(s) \geq s - 1$ for $\frac{n+1}{2} < s < n$; see also [54, prop. 15.8]. Notice that this, together with Theorem 3.2, implies Falconer's result in Theorem 3.1.

It is uncertain what should be the value of $\alpha(s)$ in general, but when we are in $\mathbb{R}^2$ this problem has been already solved. Bourgain [6] investigated it using the theory of restriction of the Fourier transform to the sphere, and the sharp value was finally settled by Wolff [73], asserting that

$$\alpha(s) = \begin{cases} s & \text{for } 0 < s < \frac{1}{2}, \\ \frac{1}{2} & \text{for } \frac{1}{2} \leq s < 1, \\ \frac{s}{2} & \text{for } 1 \leq s < 2. \end{cases}$$

As regards lower bounds for $\alpha(s)$ in general, Erdoğan [24, 25] simplified the argument of Wolff and extended it to higher dimensions. He used duality, as we did, but the bound we got from Lemma 1.1 is not, admittedly, quite a deep inequality. Erdoğan refined the bilinear theory of restriction and used clever arguments to get the improved lower bound $\alpha(s) \geq (n + 2s - 2)/4$ for $\frac{n}{2} < s < \frac{n+2}{2}$, hence by Theorem 3.2 if $E \subset \mathbb{R}^n$ has dimension $> \frac{n}{2} + \frac{1}{3}$ then $|\Delta(E)| > 0$, the best result to date.

Lucà and Rogers [50] used the refined Bourgain-Guth method to push the lower bound even further to $\alpha(s) \geq s - 1 + \frac{(n-s)^2}{(n-1)(2n-s-1)}$. Unfortunately, this bound does not improve on the state of the art of Falconer's conjecture. We can summarize the lower bounds as

$$\alpha(s) \geq \begin{cases} s & \text{for } 0 < s < \frac{n-1}{2}, \\ \frac{n-1}{2} & \text{for } \frac{n-1}{2} \leq s < \frac{n}{2}, \\ \frac{n+2s-2}{4} & \text{for } \frac{n}{2} \leq s < s_0, \\ s - 1 + \frac{(n-s)^2}{(n-1)(2n-s-1)} & \text{for } s_0 \leq s < n, \end{cases}$$

where $s_0 \sim \frac{n}{2} + \frac{2}{3} + \frac{1}{n}$ is the intersection of the lines $\frac{n+2s-2}{4}$ and $s - 1 + \frac{(n-s)^2}{(n-1)(2n-s-1)}$.

As regards upper bounds for $\alpha(s)$, Sjölin [61] used Knapp-type examples to show that $\alpha(s) \leq \frac{s}{2} + \frac{n}{2} - 1$ for $n - 2 \leq s < n$. Iosevich and Rudnev [40] show how to construct a sequence of measures $d\mu_k = \rho_k \, dx$, where ρ_k is a smooth positive function, such that $I_s(\mu_k) = 1$ and $\sigma(\mu_k)(r_k) \geq C r_k^{-\frac{n-2}{n}s-1}$ for a sequence $r_k \to \infty$; hence $\alpha(s) \leq \frac{n-2}{n}s + 1$. Essential to their argument is the count of lattice points intersecting spheres of large radius. Other proof using the same essential idea, but arguing by a duality principle, is due to Lucà and Rogers [50].

During the thesis, I proved that it is not necessary to use a sequence of measures $d\mu_k = \rho_k \, dx$ to prove that $\alpha(s) \leq \frac{n-2}{n}s + 1$; in fact, there exists a single measure decaying slowly in the whole space.

Theorem 3.3. *If $n/2 < s < n$ then there exists a measure μ with finite t-energy for $t < s$, such that $\sigma(\mu)(r_k) \geq C r_k^{-\frac{n-2}{n}s-1}$ for a sequence $r_k \to \infty$.*

Proof. We examine first the Fourier transform of a measure supported in the set (3-1), which was used by Falconer to state his conjecture. If μ is a probability measure supported in $E = E_{s/n} \times \cdots \times E_{s/n}$, then we have for the Fourier transform $|\hat{\mu}(\xi)| \geq |\int \cos(2\pi\langle \xi, x\rangle) \, d\mu(x)|$.

Since $E \subset E_{s/n,k} \times \cdots \times E_{s/n,k}$, only the points with coordinates $x_i = p_i/q_k + a_i$ for $|a_i| \le q_k^{-n/s}$ contribute to the integral. Hence for the frequencies $\xi = (N_1 q_k, \ldots, N_n q_k)$, where N_i are integers satisfying $1 \le N_i \le c q_k^{n/s-1}$, we get $\cos(2\pi\langle \xi, x\rangle) = \cos(2\pi q_k \sum N_i a_i)$, but $|2\pi q_k \sum N_i a_i| \le 2\pi n c$, so choosing c sufficiently small we get

$$|\hat{\mu}(\xi)| \ge \int \cos(2\pi\langle \xi, x\rangle)\, d\mu(x) \ge \frac{1}{2}\mu(\mathbb{R}^n) = \frac{1}{2}.$$

This is basically what we need to know about the Fourier transform of measures supported in our sets.

Assume that E is as before and that μ is a probability measure such that $I_t(\mu) < \infty$, hence $|\hat{\mu}(\xi)| \ge \frac{1}{2}$ if $\xi \in q_k\mathbb{Z}^n \cap [0, c q_k^{n/s}]^n$. Since μ is supported in a bounded set, by the uncertainty principle we can assume that $|\hat{\mu}(\xi)| \ge \frac{1}{4}$ in $\xi \in q_k\mathbb{Z}^n \cap [0, c q_k^{n/s}]^n + B(0,\rho)$, where $B(0,\rho)$ is a ball of sufficiently small radius $\rho \sim 1$. In other words, $\hat{\mu}$ concentrates around balls in the lattice $q_k\mathbb{Z}^n \cap [0, c q_k^{n/s}]^n$.

We use now a pigeonholing argument to count lattice points on spheres of certain large radius r_k, although number theoretic reasonings are also possible. The number of lattice points $\xi \in q_k\mathbb{Z}^n$ lying in the annulus $\frac{1}{10} c q_k^{n/s} \le |\xi| \le c q_k^{n/s}$ is $\sim q_k^{n(n/s-1)}$. On the other hand, for a lattice point in the annulus we have $|\xi|^2 = q_k^2(N_1^2 + \cdots + N_n^2) \in q_k^2\mathbb{Z} \cap [\frac{c^2}{100} q_k^{2n/s}, c^2 q_k^{2n/s}]$, hence the number of distinct distances from the origin to the lattice points is $\lesssim q_k^{2(n/s-1)}$. Since the points are distributed among the different distances, then we can find a sphere S_k centered at the origin and of radius $r_k \sim q_k^{n/s}$ such that the number of lattice points on it is $\gtrsim q_k^{(n-2)(n/s-1)} \sim r_k^{(n-2)(1-s/n)}$. In terms of spherical means, we have

$$\sigma(\mu)(r_k) \ge \frac{1}{16 r_k^{n-1}} \int_{S_k \cap (q_k\mathbb{Z}^n + B(0,\rho))} d\sigma \gtrsim \rho^{n-1} r_k^{(n-2)(1-s/n)-(n-1)},$$

where σ is the standard measure on S_k. We conclude thus that $\sigma(\mu)(r_k) \gtrsim r_k^{-\frac{n-2}{n}s-1}$ for a sequence $r_k \to \infty$, which is what we wanted to prove. □

Remark. It is surprising for me that the sets used by Falconer to state his conjecture do not match the best known upper bound for two and three dimensions.

If $s \le n/2$ then the fact that $|\hat{\mu}(\xi)| \ge \frac{1}{4}$ in balls around lattice points worsen the average decay to $\sigma(\mu)(r_k) \gtrsim r_k^{-s}$ for a sequence $r_k \to \infty$, because a ball can intersect many distances.

We can modify the construction of the set E in dimensions two or three to get a measure whose Fourier transform decays slower. In the case $\mathbb{R}^3$, for example, we construct first a set $E' = E_{s'/2} \times E_{s'/2}$, for $0 < s' < 2$, in the plane spanned by the first two coordinates; as above we have chosen a sequence $\{q_k\}$ increasing rapidly to define $E_{s'/2}$. Now we construct a set $E'' = E_{s''} \subset \mathbb{R}$ for $s'' = 1 - \epsilon$, using instead the sequence $l_k = \lfloor q_k^{4s''/s'} \rfloor$ to define $E_{s''}$. The dimension of $E = E' \times E''$ is $\ge s = s' + s''$ and, by similar calculations as we did before, the Fourier transform of a measure supported in E satisfies $|\hat{\mu}(\xi)| \ge \frac{1}{4}$ for

$$\xi = (\xi_1, \xi_2, \xi_3) \in (q_k\mathbb{Z}^2 \cap [0, c^{1/2} q_k^{2/s'}]^2) \times (l_k\mathbb{Z} \cap [0, c l_k^{1/s''}]) + B(0,\rho),$$

where $c > 0$ is a small constant. Let S_k be a sphere of radius[1] $r_k = cl_k^{1/s''}$ and notice that $r_k = cq_k^{4/s'}$. We see then that $|\hat{\mu}|$ concentrates on a slab $[0, r_k^{1/2}]^2 \times \{r_k\}$ of width $\rho \sim 1$. The shape of $\hat{\mu}$ may remind the reader about the classical Knapp example, hence we can think of S_k as being essentially flat at scale $r_k^{1/2}$ and then the main contribution of $|\hat{\mu}|$ to the spherical mean lies in a cap of radius $r_k^{1/2}$, consisting of all the balls $(q_k\mathbb{Z}^2 \cap [0, c^{1/2}q_k^{2/s'}]^2) \times \{r_k\} + B(0, \rho)$. By direct computation we get that $\sigma(\mu)(r_k) \gtrsim r_k^{-\frac{s+\epsilon}{2}-\frac{1}{2}}$. Since ϵ can be made arbitrarily small, we conclude that $\alpha(s) \le \frac{s}{2} + \frac{1}{2}$ for $1 < s < 3$, which coincides with the known bound. This is basically the Knapp example in disguise.

3.1 Continuous Analogue of Guth-Katz Method

I will expand on the remark made before about the relationship between Mattila's method, and Guth and Katz method. Guth and Katz used the approach outlined by Elekes and Sharir [23], which relates Erdős' distinct distances conjecture to counting intersections between helices. Guth and Katz translated this from helices to lines, and they were successful in counting intersections between lines in $\mathbb{R}^3$ by using tools from algebraic geometry, providing so a tight estimate for the least number of distinct distances determined by a set of points.

In this section we will translate Section 2 in [38] from the language of lines to tubes, however this is the furthest we can reach, as it is well-known how highly non-trivial is to pass from results for lines to tubes. Recall that if $D\delta(\mu) \in L^2$ is not zero then $D\delta(\mu)$ is a measurable function, and consequently $|\Delta(E)| > 0$. Mattila did not calculate directly $\|D\delta(\mu)\|_2$, but the quantity $\int_0^\infty D\delta(\mu)^2 \frac{dt}{t}$, which dominates $\|D\delta(\mu)\|_2^2$ for $\operatorname{supp}\mu \subset B_1$ and is related to $\sigma(\mu)$ by the Hankel transform.

Suppose that we are given a Borel set $E \subset B_1 \subset \mathbb{R}^2$ with dimension $s > 1$. To measure the size of subsets of E we use a Frostman measure μ, *i.e.* a measure that satisfies $\mu(B_r) \lesssim r^s$ for every ball B_r; we assume also that $\mu(\mathbb{R}^2) = 1$. We regularize μ using a smooth radial function $\phi \in C_0^\infty$ supported in B_1 such that $\int \phi = 1$, and we define $\mu_\rho := \mu * \phi_\rho$, where $\phi_\rho(x) = \rho^{-2}\phi(x/\rho)$. Since μ is a Frostman measure, we have that $\|\mu_\rho\|_\infty \lesssim \rho^{s-2}$. The induced measure on $\Delta(E)$ is $\delta(\mu_\rho)(F) = \int \mathbb{1}_F(|x-y|)\, d\mu_\rho(x) d\mu_\rho(y)$ for $F \subset \mathbb{R}$.

We would like to show that $\|D\delta(\mu_\rho)\|_2 \le C$ for some constant independent of ρ. If this is the case, then by the weak compactness of bounded sets in L^2, there is a sequence $\rho_k \to 0$ and a function $g \in L^2$ such that $D\delta(\mu_{\rho_k}) \rightharpoonup g$ in L^2. Moreover, it is possible to prove that $\int f D\delta(\mu_\rho) \to \int f D\delta(\mu) = \int f(|\cdot|) * \mu(y)\, d\mu(y)$ for every continuous function f, hence $D\delta(\mu) = g \in L^2$. To check that $g \neq 0$, notice that $1 = \int D\delta(\mu_{\rho_k}) = \int_{[0,2]} D\delta(\mu_{\rho_k}) \to \int_{[0,2]} g$, because $\mathbb{1}_{[0,2]} \in L^2$. The same argument is equally valid if we replace L^2 by any L^p for $p > 1$.

Since μ_ρ is smooth, the measure $D\delta(\mu_\rho)$ is also smooth, except possibly for the origin. The derivative of $\delta(\mu_\rho)$ can be calculated as $D\delta(\mu_\rho)(t) = \lim_{r\to 0^+} \frac{1}{r}\delta(\mu_\rho)([t, t+r))$, and for $t = 0$ we have $\frac{1}{r}\delta(\mu_\rho)([0,r)) \lesssim \rho^{s-2} r \to 0$; moreover, it is not hard to see that $D\delta(\mu_\rho)(t) \lesssim_\rho t$,

[1] More precisely, we should choose $\lfloor cl_k^{1/s''-1} \rfloor l_k$.

hence the function $\frac{1}{t}D\delta(\mu_\rho)^2$ is integrable. Since $\frac{1}{r}\delta(\mu_\rho)([t,t+r)) \lesssim_\rho (2t+r)$, we can use dominated convergence and Fubini to show that

$$\int_0^\infty D\delta(\mu_\rho)^2\frac{dt}{t} = \lim_{r\to 0^+}\frac{1}{r^2}\int_r^\infty \mu_\rho * \mathbb{1}_{\{|z|\in[t,t+r)\}}(x_1)\mu_\rho * \mathbb{1}_{\{|z|\in[t,t+r)\}}(x_2)\,d\mu_\rho(x_1)d\mu_\rho(x_2)\,\frac{dt}{t}$$
$$= \lim_{r\to 0^+}\int\Big(\frac{1}{r^2}\int_r^\infty \mathbb{1}_{[t,t+r)}(|x_1-y_1|)\mathbb{1}_{[t,t+r)}(|x_2-y_2|)\frac{dt}{t}\Big)$$
$$d\mu_\rho(y_1)d\mu_\rho(y_2)d\mu_\rho(x_1)d\mu_\rho(x_2). \quad \text{(3-9)}$$

By the change of variables $x_i - y_i = l_i(\cos\theta_i,\sin\theta_i)$ and $\frac{1}{2}(x_i+y_i) = u_i$, for $i=1,2$, we have

$$\int_0^\infty D\delta(\mu_\rho)^2\frac{dt}{t} = \lim_{r\to 0^+}\int l_1l_2H_r(l_1,l_2)$$
$$\Big(\int \mu_\rho(u_1+\frac{x_1-y_1}{2})\mu_\rho(u_1-\frac{x_1-y_1}{2})\mu_\rho(u_2+\frac{x_2-y_2}{2})\mu_\rho(u_2+\frac{x_2-y_2}{2})\,d\theta du\Big)dl, \quad \text{(3-10)}$$

where H_r is the term inside parentheses in (3-9). We need the next lemma.

Lemma 3.4. *If f is a continuous and compactly supported function in $\mathbb{R}^2_{\geq 0}$ such that $|f(l)| \lesssim |l|^\beta$, for $\beta > 0$ and $|l| \ll 1$, then*

$$\lim_{r\to 0^+}\int H_r(l)f(l)\,dl = \int \delta(l_1-l_2)f(l)\frac{dl}{l_2}. \quad \text{(3-11)}$$

Proof. If $0 \leq l_2 - l_1 < r$ and $l_1, l_2 \geq r$ then we have

$$H_r^+(l_1,l_2) := \frac{1}{r^2}\int_r^\infty \mathbb{1}_{[t,t+r)}(l_1)\mathbb{1}_{[t,t+r)}(l_2)\frac{dt}{t} = \frac{1}{r^2}\int_{\max\{r,l_2-r\}}^{l_1}\frac{dt}{t},$$

otherwise the integral is zero, *i.e.* $H_r^+(l_1,l_2) := 0$ for $l_2 - l_1 \geq r$; we extend H_r^+ to $l_2 < l_1$ by setting $H_r^+(l_1,l_2) = 0$. We define $H_r^-(l_1,l_2) := H_r^+(l_2,l_1)$ and observe that $H_r = H_r^+ + H_r^-$. Now we write

$$\int H_r^+(l)f(l)\,dl = \frac{1}{r^2}\int_r^\infty\int_r^\infty \mathbb{1}_{\{0\leq l_2-l_1\leq r\}}\int_{\max\{r,l_2-r\}}^{l_1} f(l)\frac{dt}{t}dl$$
$$= \frac{1}{r^2}\int_r^{2r}\int_r^{l_2} f(l)\int_r^{l_1}\frac{dt}{t}dl_1dl_2 + \frac{1}{r^2}\int_{2r}^\infty\int_{l_2-r}^{l_2} f(l)\int_{l_2-r}^{l_1}\frac{dt}{t}dl_1dl_2 = A+B.$$

For the first term we have, for $r<1$, that

$$|A| \lesssim \frac{\log r}{r^2}\int_r^{2r}\int_r^{2r}|l|^\beta\,dl_1dl_2 \lesssim r^\beta\log r \xrightarrow{r\to 0^+} 0.$$

For the second term, consider the function $h_r(l_2) = r^{-2}\mathbb{1}_{\{l_2>2r\}}\int_{l_2-r}^{l_2} f(l)\int_{l_2-r}^{l_1}\frac{dt}{t}dl_1$ that converges pointwise to $\frac{1}{2l_2}f(l_2,l_2)$, and

$$|h_r(l_2)| \leq \mathbb{1}_{\{l_2>2r\}}\frac{1}{r}\int_{l_2-r}^{l_2} f(l)\frac{dl_1}{l_2-r} \lesssim l_2^{\beta-1}, \quad \text{for } l_2, r \ll 1;$$

hence the function $h_r(l_2)$ is dominated by an integrable function and we have that

$$\frac{1}{r^2}\int_{2r}^{\infty}\int_{l_2-r}^{l_2} f(l)\int_{l_2-r}^{l_1}\frac{dt}{t}dl_1 dl_2 \xrightarrow{r\to 0^+} \frac{1}{2}\int \delta(l_1-l_2)f(l)\frac{dl}{l_2}.$$

A similar result holds for H_r^-, concluding the lemma. □

In (3-10) the term in parentheses is a bounded function on l, so we can apply the preceding Lemma with $\beta = 2$ and return to the original variables to get

$$\int_0^{\infty} D\delta(\mu_\rho)^2\frac{dt}{t} = \int \frac{1}{|x_1-y_1|}\delta(|x_1-y_1|-|x_2-y_2|)\, d\mu_\rho(y_1)d\mu_\rho(y_2)d\mu_\rho(x_1)d\mu_\rho(x_2).$$

The term $\delta(|x_1-y_1|-|x_2-y_2|)$ counts over quadruples, that is the set of points $(x_1,y_1,x_2,y_2)\in\mathbb{R}^8$ such that $|x_1-y_1|=|x_2-y_2|$. This set is denoted as $Q(P)$ by Guth and Katz, and the cardinality of this set is related to the number of distinct distances as we have just done, but replacing integrals by sums; see Lemma 2.1 in [38].

Following Guth and Katz, we want to relate the number of quadruples to the size of $E\cap gE$, where g is a positively-oriented isometry of the plane. Hence we define the function

$$\nu_\rho(g) := \int \mu_\rho(x)\mu_\rho(g^{-1}x)\, dx.$$

Every isometry can be expressed as $g=\tau_h R_\theta$, where $\tau_h(x)=x+h$ is a translation and R_θ is a rotation by angle θ, hence we can parametrize the set of positively-oriented isometries G by $\mathbb{R}^2\times S^1$.

Motivated by the identity (2.2) in [38], we calculate the L^2 norm of ν_ρ to get, after a series of exchange of integrals and changes of variables,

$$\begin{aligned}
\int |\nu_\rho(g)|^2\, dg &= \int \mu_\rho(x_1)\mu_\rho(R_\theta^{-1}(x_1-h))\mu_\rho(x_2)\mu_\rho(R_\theta^{-1}(x_2-h))\, dx_1dx_2dhd\theta \\
&= \int \mu_\rho(x_1)\mu_\rho(x_2)\Big(\int \mu_\rho(R_\theta^{-1}(x_1-h))\mu_\rho(R_\theta^{-1}(x_2-h))\, dhd\theta\Big)\, dx_1dx_2 \\
&= \int \mu_\rho(x_1)\mu_\rho(x_2)\Big(\int \mu_\rho(h+R_\theta^{-1}(x_1-x_2))\mu_\rho(h)\, dhd\theta\Big)\, dx_1dx_2 \\
&= \int \mu_\rho(x_1)\mu_\rho(x_2)\mu_\rho(h)\Big(\int \mu_\rho(h+R_\theta^{-1}(x_1-x_2))\, d\theta\Big)\, dx_1dx_2dh \\
&= \int \mu_\rho(x_1)\mu_\rho(x_2)\mu_\rho(h)\frac{1}{|x_1-x_2|}\int \mu_\rho(h+z)\delta(|z|-|x_1-x_2|)\, dz\, dx_1dx_2dh \\
&= \int_0^{\infty} D\delta_\rho^2\frac{dt}{t}.
\end{aligned}$$

Therefore, Mattila's strategy is equivalent to obtaining a uniform control of the quantity $\|\nu_\rho\|_2$ with respect to ρ.

Now, it is necessary to express ν_ρ in terms of the curves $S_{xy} := \{g \in G \mid gx = y\}$ in $\mathbb{R}^2 \times S^1$. This follows from

$$\begin{aligned}\nu_\rho(g) &= \int \Big(\int \phi_\rho(z-y)\phi_\rho(g^{-1}z - x)\, dz \Big) d\mu(x) d\mu(y) \\ &= \int \Big(\int \phi_\rho(z)\phi_\rho(z + g^{-1}y - x)\, dz \Big) d\mu(x) d\mu(y),\end{aligned}$$

where we used the radial symmetry of ϕ_ρ. The term in parentheses vanishes unless g belongs to the set $\{g \in G \mid |gx - y| < 2\rho\}$. If we define

$$S_{xy}(r) := \bigcup_{(x',y') \in Q_r(x) \times Q_r(y)} S_{x'y'},$$

where $Q_r(x)$ is the cube of side length r centered at x, then $\{g \mid |gx - y| < 2\rho\} \subset S_{xy}(5\rho)$. Hence, we get

$$\nu_\rho(g) \le c \int \frac{1}{\rho^2} \mathbb{1}_{S_{xy}(5\rho)}(g)\, d\mu(x) d\mu(y), \tag{3-12}$$

and ν_ρ can be seen as the weighted union of fat curves $S_{xy}(5\rho)$; see Lemma 2.6 in [38].

Guth and Katz applied a transformation that carries the helices $S_{x,y}$ into lines, but if we naively apply the same transformation, then we would get unpleasant consequences, and instead of tubes we would get “oars”, that is, a tube with a cone attached to each end. Hence, we depart slightly from Guth and Katz, and split the norm $\|\nu_\rho\|_2$ as

$$\int |\nu_\rho(g)|^2\, dg = \int_{\theta \in (\pi/2, 3\pi/2)} |\nu_\rho(g)|^2\, dg + \int_{\theta \in (-\pi/2, \pi/2)} |\nu_\rho(g)|^2\, dg \tag{3-13}$$

To estimate the first term, we use the same transformation applied by Guth and Katz, that is, we parametrize the elements in G by their fixed point z and the angle of rotation around z; this is possible because we are restricted to $\theta \in (\pi/2, 3\pi/2)$. The unique fixed point of $g = \tau_h R_\theta$ is

$$z = \frac{1}{2} \begin{pmatrix} 1 & -\cot\frac{\theta}{2} \\ \cot\frac{\theta}{2} & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

and the angle of rotation around z coincides with θ. Hence, we define the change of variables

$$z_1 = \frac{1}{2} h_1 - \frac{1}{2}\cot\frac{\theta}{2} h_2, \qquad z_2 = \frac{1}{2}\cot\frac{\theta}{2} h_1 + \frac{1}{2} h_2 \qquad \text{and} \qquad t = \cot\frac{\theta}{2}.$$

The first term in (3-13) transforms into

$$\int_{\theta \in (\pi/2, 3\pi/2)} |\nu_\rho(h,\theta)|^2\, dh d\theta = 8 \int_{-1}^{1} |\nu_\rho(z,t)|^2 \frac{dzdt}{(1+t^2)^2}.$$

A curve S_{xy} becomes a line

$$l_{x,y} = \{(a + vt, t) \mid a = \frac{x+y}{2} \text{ and } v = \frac{(x-y)^\perp}{2}\}$$

Here, if $w = (w_1, w_2)$ then $w^\perp = (w_2, -w_1)$. Therefore, the fat curve $S_{xy}(5\rho)$ becomes the tube

$$\bigcup_{(x',y')\in Q_\rho(x)\times Q_\rho(y)} l_{x'y'} \subset T^\rho_{x,y} := \{(a + tv + u, t) \mid u \in B_{10\rho},\ a = \frac{x+y}{2} \text{ and } v = \frac{(x-y)^\perp}{2}\}. \tag{3-14}$$

Replacing this in (3-12) and discretizing the integral at scale ρ we have

$$\int_{\theta\in(\pi/2,3\pi/2)} |\nu_\rho(g)|^2\, dg \leq \frac{C}{\rho^4}\int_{-1}^{1}\int \Big| \sum_{x,y\in\rho\mathbb{Z}^2} \mathbb{1}_{T^\rho_{x,y}}(z,t)\mu(Q_\rho(x))\mu(Q_\rho(y))\Big|^2\, dzdt;$$

recall that $Q_\rho(x)$ is a cube of side length ρ centered at x.

To evaluate the second term in (3-13) we use, instead of the fixed point, the inverse fixed point $gz = -z$. We repeat the previous procedure with the new change of variables

$$z_1 = -\frac{1}{2}h_1 - \frac{1}{2}\tan\frac{\theta}{2}h_2, \qquad z_2 = \frac{1}{2}\tan\frac{\theta}{2}h_1 - \frac{1}{2}h_2 \qquad \text{and} \qquad t = \tan\frac{\theta}{2}$$

to get

$$\int_{\theta\in(-\pi/2,\pi/2)} |\nu_\rho(g)|^2\, dg \leq \frac{C}{\rho^4}\int_{-1}^{1}\int \Big| \sum_{x,y\in\rho\mathbb{Z}^2} \mathbb{1}_{\tilde{T}^\rho_{x,y}}(z,t)\mu(Q_\rho(x))\mu(Q_\rho(y))\Big|^2\, dzdt,$$

where

$$\tilde{T}^\rho_{x,y} := \{(a + tv + u, t) \mid u \in B_{10\rho},\ a = \frac{x-y}{2} \text{ and } v = -\frac{(x+y)^\perp}{2}\}. \tag{3-15}$$

We can state thus the following theorem.

Theorem 3.5. *If for every measure supported in B_1 and satisfying the conditions $\mu(\mathbb{R}^2) = 1$ and $\mu(B_\rho) \lesssim \rho^s$ for every $\rho < 1$, we have the inequality*

$$\Big\| \sum_{x,y\in\rho\mathbb{Z}^2} \frac{1}{|T^\rho_{x,y}|}\mathbb{1}_{T^\rho_{x,y}}\mu(Q_\rho(x))\mu(Q_\rho(y))\Big\|_{L^2(\mathbb{R}^2\times[-1,1])} \lesssim C_\mu, \tag{3-16}$$

where C_μ does not depend on $\rho < 1$, and the tubes are given by (3-14) *or* (3-15), *then Falconer's conjecture holds for dim$(E) > s$.*

The inequality (3-16) holds for $s = 2$; in fact, through every point pass $\lesssim \rho^{-2}$ tubes, hence $\sum_{x,y\in\Lambda_\rho} \frac{1}{|T^\rho_{x,y}|}\mathbb{1}_{T^\rho_{x,y}}\mu(Q_\rho(x))\mu(Q_\rho(y)) \lesssim 1$.

4 Sharp Dependence on Transversality in Higher Dimensions

In this chapter we continue our discussion on sharp dependence on transversality, but for the paraboloid in higher dimensions. While I was unable to make the argument to work in higher dimensions, I wish to write some partial progress.

We provide a large supply of "standard"n-tuples where the multiscale analysis is efficient, and they are constructed systematically in the first section. In the second section, we repeat basically our arguments in Section 2.3, to verify that the induction on scales is indeed efficient. To avoid technical complications, we define the extension operator over the hypersurface $S = \{(\xi, -\frac{1}{2}|\xi|^2) \mid \xi \in B_1\}$.

The main result of this chapter is Theorem 4.8. I know neither how one can cover an arbitrary transversal n-tuple of sets efficiently with the standard n-tuples of sets constructed in this chapter, nor how one can prove the first step of the induction on scales for all standard n-tuples. The content here is rather technical in nature and perhaps of unnecessary generality, nevertheless I wanted to include it as a reference, which I hope will help to solve the problem in full.

4.1 Standard Sets

The philosophy of this section is relatively simple: to find n-tuples of sets $\{S_k\}$ with the corresponding decomposition into caps α_k, for $k = 1, \dots, n$, such that for every n caps $\alpha_k \subset S_k$ the functions $\mathcal{E}\mathbb{1}_{\alpha_k}$, which look like tubes, intersect essentially in a fixed set, which is an ellipsoid, and the functions $f_k = \mathbb{1}_{\alpha_k}$ realize the sharp constant in the conjectured multilinear inequality

$$\int \prod_{k=1}^{n} |\mathcal{E} f_k|^{\frac{2}{n-1}} \, d\xi \leq C\theta^{-\frac{1}{n-1}} \prod_{k=1}^{n} \|f_k\|_2^{\frac{2}{n-1}}.$$

The work of Ramos [60] suggests that this is a well-behaved decomposition under changes of scale, as we will see in the next section.

4.1.1 Functions attaining sharp transversality

Our first aim is to find subsets of the paraboloid α_k such that the functions $\mathbb{1}_{\alpha_k}$ realize the sharp constant in the multilinear inequality. First, we need to describe n-tuples of "tubes"$\{T_k\}_{k=1}^n$ with elliptic transversal sections E_k, and with direction vectors $\{(v_k,1)\}_{k=1}^n$, for $v_k \in \mathbb{R}^{n-1}$, attaining almost equality in the Kakeya multilinear inequality

$$\int |\prod_{k=1}^n \mathbb{1}_{T_k}|^{\frac{1}{n-1}} \leq C\theta^{-\frac{1}{n-1}} \prod_{k=1}^n |E_k|^{\frac{1}{n-1}}. \tag{4-1}$$

Since this inequality is a change of variables of the Loomis-Whitney inequality, it suffices to consider optimal examples for

$$\|\prod_k f_k \circ \pi_k\|_{L^{1/(n-1)}(\mathbb{R}^n)} \leq \prod_k \|f_k\|_{L^1(\mathbb{R}^{n-1})}. \tag{4-2}$$

This becomes almost an identity if we replace f_k by $\mathbb{1}_{B_1}$, so that we have n tubes

$$T_k := \mathbb{1}_{B_1} \circ \pi_k = \Big\{x \in \mathbb{R}^n \mid x_1^2 + \cdots + \widehat{x_k^2} + \cdots + x_n^2 \leq 1\Big\} \quad \text{(the symbol under } \widehat{\cdot} \text{ is removed).}$$

The tubes intersect essentially in a ball of radius 1, to be precise, $B_1 \subset \bigcap_k T_k \subset B_{c_n}$ for $c_n = \sqrt{n/(n-1)}$. We can provide more examples of tubes running parallel to the coordinate axes attaining almost the identity in (4-2), by applying to $\mathbb{R}^n$ the transformation

$$D_{\boldsymbol{\lambda}} := \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where $\boldsymbol{\lambda}$ denotes the vector $(\lambda_1, \cdots, \lambda_n)$; hence, our transformed tubes $T_{\boldsymbol{\lambda},k} = D_{\boldsymbol{\lambda}} T_k$ now have elliptic section and can be written as

$$T_{\boldsymbol{\lambda},k} = \Big\{x \in \mathbb{R}^n \mid \frac{x_1^2}{\lambda_1^2} + \cdots + \widehat{\frac{x_k^2}{\lambda_k^2}} + \cdots + \frac{x_n^2}{\lambda_n^2} \leq 1\Big\}.$$

If we are given n vectors $((v_1,1), \ldots, (v_n,1))$, for $v_k = (v_{1,k}, \ldots, v_{n-1,k}) \in B_1$, then examples of tubes pointing in directions $\{(v_k,1)\}$ and attaining almost the identity in (4-1) are given by $\Omega T_{\boldsymbol{\lambda},k}$, where

$$\Omega := \begin{pmatrix} v_1 & \cdots & v_n \\ 1 & \cdots & 1 \end{pmatrix}.$$

To get an explicit description of the tubes $\Omega T_{\boldsymbol{\lambda},k}$, we note that $\Omega z = (\Omega_k \pi_k z + t v_k, t)$, where $t = z_1 + \cdots + z_n$ and Ω_k is the $(n-1) \times (n-1)$ matrix

$$\Omega_k = (v_1 - v_k \cdots v_n - v_k) = \begin{pmatrix} v_{1,1} - v_{1,k} & \cdots & v_{1,n} - v_{1,k} \\ \vdots & & \vdots \\ v_{n-1,1} - v_{n-1,k} & \cdots & v_{n-1,n} - v_{n-1,k} \end{pmatrix}.$$

Thus, if we define $D_k = \operatorname{diag}(\lambda_1 \cdots \widehat{\lambda}_k \cdots \lambda_n)$, then the tubes are given by

$$\Omega T_{\boldsymbol{\lambda},k} = \{(x', x_n) = (y + x_n v_k, x_n) \mid y \in \Omega_k D_k B_1\};$$

thinking of x_n as the time variable, $\Omega T_{\boldsymbol{\lambda},k}$ is generated by the ellipse $\Omega_k D_k B_1$ traveling with velocity v_k. The new tubes intersect essentially in the ellipsoid $\Omega D_{\boldsymbol{\lambda}} B_1$, which has volume $\sim \prod_k \lambda_k |\det \Omega|$.

The tube $\Omega T_{\boldsymbol{\lambda},k}$ can be seen as the support of a wave-packet associated to some suitable cap α_k. To identify how are those caps, let us see what is the result of applying the extension operator to the function $f = \mathbb{1}_{\delta Q B_1 + v}$, where Q is a linear operator, $\delta > 0$ and $v \in B_1$. We get

$$\begin{aligned}\mathcal{E}f(\xi) &= \int_{\delta Q B_1 + v} e(-\langle x', \xi\rangle + \frac{1}{2} x_n |\xi|^2)\, d\xi \\ &= \delta^{n-1} |\det Q| e(-\langle x', v\rangle + \frac{1}{2} x_n |v|^2) \int_{B_1} e(-\langle \delta Q^t (x' - x_n v), \xi\rangle + \frac{1}{2}\delta^2 x_n |Q\xi|^2)\, d\xi.\end{aligned}$$

Setting $y' = \delta Q^t(x' - x_n v)$ and $y_n = \delta^2 x_n$, it suffices to calculate

$$I_Q = \int_{B_1} e(-\langle y', \xi\rangle + \frac{1}{2} y_n |Q\xi|^2)\, d\xi.$$

Since $\sup_{|\xi|\le 1} |Q\xi|^2 = |Q|^2$, where $|Q|$ is the $\ell^2 \to \ell^2$ norm operator, the integral I_Q is the Fourier transform of a paraboloid contained in $B_1 \times [-|Q|^2, |Q|^2]$, hence $|I_Q| \sim 1$ in the set of points $y \in B_1 \times [-\frac{1}{|Q|^2}, \frac{1}{|Q|^2}]$; moreover, the integral decays faster than any polynomial for $|y'| \to \infty$. Restoring the original variables x, we get that $\mathcal{E}f$ is essentially supported in the tube

$$T_{\delta Q, v} = \{(x', x_n) = (z + x_n v, x_n) \mid z \in \delta^{-1} Q^{-t} B_1 \text{ and } |x_n| \le \frac{\delta^{-2}}{|Q|^2}\},$$

and in this tube $|\mathcal{E}f| \sim \delta^{n-1} |\det Q|$. We can think of $T_{\delta Q, v}$ as the tube generated by moving the ellipse $\delta^{-1} Q^{-t} B_1$ with velocity v.

We compare $\Omega T_{\boldsymbol{\lambda},k}$ and $T_{\delta Q, v}$ to find out that $Q = \Omega_k^{-t} D_k^{-1}$ and $v = v_k$ generate the tube $\delta^{-1} \Omega T_{\boldsymbol{\lambda},k}$, whenever $|x_n| \le \frac{\delta^{-2}}{|Q|^2}$, by means of the function $f_k = \mathbb{1}_{\delta \Omega_k^{-t} D_k^{-1} B_1 + v_k}$ after applying the extension operator to it; moreover, if δ is small enough, and we will determine how small it should be later in Lemma 4.3, then they intersect essentially in the region $\delta^{-1} \Omega D_{\boldsymbol{\lambda}} B_1$ with volume $\sim \delta^{-n} \prod_k \lambda_k |\det \Omega| \sim \delta^{-n} \prod_k \lambda_k \theta$. Hence, we replace f_k in the multilinear extension inequality to reach

$$\delta^n \prod_k \lambda_k^{-1} \theta^{-\frac{n+1}{n-1}} \sim \int |\prod_k \mathcal{E} f_k|^{\frac{2}{n-1}}\, d\xi \le C \prod_k \|f_k\|_2^{\frac{2}{n-1}} \sim C \delta^n \prod_k \lambda_k^{-1} \theta^{-\frac{n}{n-1}},$$

therefore $C \gtrsim \theta^{-\frac{1}{n-1}}$. We have concluded so our first aim, to find caps α_k such that the functions $\mathbb{1}_{\alpha_k}$ attain the conjectured sharp constant in the multilinear inequality.

4.1.2 Regions of stability: standard n-tuples

Once we have suitable caps at hand, we want to know how much we can perturb the vectors $\{v_k\}_{k=1}^n$, or what amounts to the same, how much we can perturb Ω, so that the set of intersection of the tubes $\Omega T_{\boldsymbol{\lambda},k}$ preserves the form. Since the tubes intersect in the ellipsoid $\Omega D_{\boldsymbol{\lambda}} B_1$, we want to find neighborhoods S_k of the points $\{v_k\}_{k=1}^n$ so that the collection of tubes $\Omega' T_{\boldsymbol{\lambda},k}$, where Ω' is defined by vectors $v_k' \in S_k$, intersect essentially in the same ellipsoid, in other words, we want to ensure that $\frac{1}{2}\Omega D_{\boldsymbol{\lambda}} B_1 \subset \Omega' D_{\boldsymbol{\lambda}} B_1 \subset 2\Omega D_{\boldsymbol{\lambda}} B_1$. These regions of stability are what we call standard n-tuples.

Let us define $W = \Omega^{-1}\Omega'$ ($|\det W| \sim 1$) and we impose on the perturbation Ω' the condition

$$D_{\boldsymbol{\lambda}}^{-1} W D_{\boldsymbol{\lambda}} = I + U, \tag{4-3}$$

where $U = (u_1 \cdots u_n)$ is a matrix satisfying $|u_k| \le c_n \ll 1$ for every k. Since $|U| = |U|_{\ell^2 \to \ell^2} \le \sqrt{n} c_n$, we can choose $c_n = n^{-1/2}\frac{1}{10}$ to be completely certain that $\frac{1}{2}B_1 \subset (I+U)B_1 \subset 2B_1$. To translate the perturbations Ω' into neighborhoods of the vectors v_k, we write

$$\Omega' = \Omega W = \Omega + \Omega D_{\boldsymbol{\lambda}} U D_{\boldsymbol{\lambda}}^{-1} = \Omega + (\lambda_1^{-1}\Omega D_{\boldsymbol{\lambda}} u_1 \cdots \lambda_n^{-1}\Omega D_{\boldsymbol{\lambda}} u_n).$$

Since $(v_k', 1) = (v_k, 1) + \lambda_k^{-1}\Omega D_{\boldsymbol{\lambda}} u_k$, necessarily the last coordinate of $\Omega D_{\boldsymbol{\lambda}} u_k$ is zero, hence $\langle \boldsymbol{\lambda}, u_k\rangle = 0$. Using this condition it is not hard to see that $\lambda_k^{-1}\Omega D_{\boldsymbol{\lambda}} u_k = \lambda_k^{-1}\iota_n \Omega_k D_k \pi_k u_k$, where ι_n is the inclusion $\iota_n(x) = (x_1, \ldots, x_{n-1}, 0)$. We get thus the neighborhoods of stability $v_k + V_{k,\boldsymbol{\lambda}} := v_k + c_n \lambda_k^{-1}\Omega_k D_k \pi_k (B_1 \cap H_{\boldsymbol{\lambda}})$, where $B_1 \subset \mathbb{R}^n$ and $H_{\boldsymbol{\lambda}}$ is the hyperplane $\langle \boldsymbol{\lambda}, u\rangle = 0$. Notice that all the sets $V_{k,\boldsymbol{\lambda}}$ are equal, up to the factor λ_k^{-1}, therefore to simplify computations we choose k^* such that $\lambda_{k^*} = \max\{\lambda_k\}$, so that $\frac{\lambda_{k^*}}{|\boldsymbol{\lambda}|} B_1 \subset \pi_{k^*}(B_1 \cap H_{\boldsymbol{\lambda}}) \subset B_1 \subset \mathbb{R}^{n-1}$, hence our neighborhoods look similar to $\lambda_k^{-1}\Omega_{k^*} D_{k^*} B_1$. Since $n^{-1/2} \le \frac{\lambda_{k^*}}{|\boldsymbol{\lambda}|}$ and $c_n \le \frac{1}{10\sqrt{n}}$, we can write the definition.

Definition 4.1. A standard $(n, \boldsymbol{\lambda})$-tuple is the n-tuple of sets

$$v_k + V_{k,\boldsymbol{\lambda}} := v_k + \frac{1}{10 n \lambda_k}\Omega_{k^*} D_{k^*} B_1, \qquad \text{for } k = 1, \ldots, n, \tag{4-4}$$

where $\lambda_{k^*} = \max\{\lambda_k\}$.

4.1.3 Decomposition of standard n-tuples

Once we have defined the standard n-tuples, we must write its decomposition into caps. We know that the caps centered at v_k are $\delta \Omega_k^{-t} D_k^{-1} B_1$, but how can we define consistently the caps centered at other points $v_k' \in v_k + V_{k,\boldsymbol{\lambda}}$? The definition of a cap centered at $v_k' \in v_k + V_{k,\boldsymbol{\lambda}}$ is different if we complement v_k' with different groups of vectors $v_j', v_j'' \in v_j + V_{j,\boldsymbol{\lambda}}$, for $j \ne k$, to get different matrices Ω' and Ω''. We must thus get a consistent decomposition into caps and verify that the wave-packets intersect in the fixed set $\Omega D_{\boldsymbol{\lambda}} B_1$. It is natural then to study the stability of the caps, which is the same as studying the structure of the operator

$D_k^{-1}W_kD_k : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, where $W_k := \Omega_k^{-1}\Omega_k'$. The next lemma is a detailed description of these operators, and can be skipped without serious consequences.

Theorem 4.2. *We define the matrix U_k by the identity*

$$D_k^{-1}W_kD_k = I + U_k. \tag{4-5}$$

If we choose an orthonormal basis $f_1, \cdots, f_{n-2}, f'_{n-1}$, where $f'_{n-1} = |\pi_k\boldsymbol{\lambda}|^{-1}(\lambda_1, \cdots, \hat{\lambda}_k, \cdots, \lambda_n)$, then there are vectors $b_j = (b_{1,j}, \cdots, b_{n-1,j})$, for $j = 1, \cdots, n$, such that the operator $B : \mathbb{R}^n \to \mathbb{R}^{n-1}$, represented by the matrix $(b_1 \cdots b_n)$, satisfies $|B|_{\ell^2\to\ell^2} \le \frac{1}{10}$, and the matrix of U_k in the basis $f_1, \cdots, f_{n-2}, f'_{n-1}$ is

$$U_k = \begin{pmatrix} b_{1,1} & \cdots & b_{1,n-2} & b_{1,n-1} - b_{1,n}\frac{|\pi_k\boldsymbol{\lambda}|}{\lambda_k} \\ \vdots & & \vdots & \vdots \\ b_{n-2,1} & \cdots & b_{n-2,n-2} & b_{n-2,n-1} - b_{n-2,n}\frac{|\pi_k\boldsymbol{\lambda}|}{\lambda_k} \\ b_{n-1,1}\frac{\lambda_k}{|\boldsymbol{\lambda}|} & \cdots & b_{n-1,k}\frac{\lambda_k}{|\boldsymbol{\lambda}|} & b_{n-1,n-1}\frac{\lambda_k}{|\boldsymbol{\lambda}|} - b_{n-1,n}\frac{|\pi_k\boldsymbol{\lambda}|}{|\boldsymbol{\lambda}|} \end{pmatrix}. \tag{4-6}$$

Moreover, if all the columns in U (see (4-3)) vanish except possibly u_k, then all the vectors b_j also vanish for $1 \le j \le n-1$. On the other hand, if $u_k = 0$ then $b_n = 0$.

Remark. If all the columns in U vanish except possibly u_k, then it means that given n vectors $\{v_j\}_{j=1,\ldots,n}$, we let all the vectors fixed except possibly v_k. In this case, if $|\pi_k\boldsymbol{\lambda}|/\lambda_k$ is large, and this can happen, then both caps $v_k + \delta\Omega_k^{-t}D_k^{-1}B_1$ and $v'_k + \delta\Omega_k'^{-t}D_k^{-1}B_1$ will be quite different each other. Therefore, we have an instability result.

The condition $u_k = 0$ means that given n vectors $\{v_j\}_{j=1,\ldots,n}$, we let all them free to move, except v_k. In this case, $|U_k| \le \frac{1}{10}$ and the cap $v_k + \delta\Omega_k^{-t}D_k^{-1}B_1$ is stable, which suggests the possibility of a meaningful decomposition into caps, in spite of its possible large variation when moving v_k, as remarked in the previous paragraph.

Proof. If x satisfies $\langle\boldsymbol{\lambda}, x\rangle = 0$, then

$$\Omega' D_{\boldsymbol{\lambda}} x = \begin{pmatrix} \Omega'_k D_k \pi_k x \\ 0 \end{pmatrix}, \qquad \text{for every } k = 1, \ldots, n.$$

Using (4-3) we get

$$\Omega D_{\boldsymbol{\lambda}}(I+U)x = \begin{pmatrix} \Omega'_k D_k \pi_k x \\ 0 \end{pmatrix}.$$

Since $\langle\boldsymbol{\lambda}, (I+U)x\rangle = 0$ (recall that $\langle u_j, \boldsymbol{\lambda}\rangle = 0$), we have $\Omega'_k D_k \pi_k = \Omega_k D_k \pi_k (I+U)$. The projection π_k turns into a bijection between $H_{\boldsymbol{\lambda}} = \{x \mid \langle x, \boldsymbol{\lambda}\rangle = 0\}$ and $\{x_k = 0\}$, hence we can write

$$D_k^{-1}W_kD_k = D_k^{-1}\Omega_k^{-1}\Omega'_kD_k = \pi_k(I+U)\pi_k^{-1},$$

and we see that $U_k = \pi_k U \pi_k^{-1}$.

We choose an orthonormal basis $f_1, \dots, f_{n-2}$ of $\{x_k = 0\} \cap H_{\boldsymbol{\lambda}}$, so by adding the vector $f'_{n-1} = \pi_k \boldsymbol{\lambda}/|\pi_k \boldsymbol{\lambda}|$ we get an orthonormal basis of $\{x_k = 0\}$; on the other hand, by adding $f_{n-1} = \frac{\lambda_k}{|\boldsymbol{\lambda}|}(\frac{\lambda_1}{|\pi_k \boldsymbol{\lambda}|}, \dots, -\frac{|\pi_k \boldsymbol{\lambda}|}{\lambda_k}, \dots, \frac{\lambda_n}{|\pi_k \boldsymbol{\lambda}|})$ we get an orthonormal basis of $H_{\boldsymbol{\lambda}}$. If we write $x = x_1 f_1 + \cdots + x_{n-1} f_{n-1}$, then it is clear that $\pi_k x = x_1 f_1 + \cdots + x_{n-2} f_{n-2} + x_{n-1} \frac{\lambda_k}{|\boldsymbol{\lambda}|} f'_{n-1}$, and if we write $z = z_1 f_1 + \cdots + z_{n-2} f_{n-2} + z_{n-1} f'_{n-1}$, then $\pi_k^{-1} z = z_1 f_1 + \cdots + z_{n-2} f_{n-2} + z_{n-1} \frac{|\boldsymbol{\lambda}|}{\lambda_k} f_{n-1}$.

To write the matrix of U_k with respect to the basis $f_1, \cdots, f_{n-2}, f'_{n-1}$ we look at the action of U_k over each element. Notice that $U : \mathbb{R}^n \to H_{\boldsymbol{\lambda}}$, so for $1 \le i \le n-2$ we get

$$f_i \mapsto \pi_k U \pi_k^{-1} f_i = \pi_k U f_i = \pi_k (b_{1,i} f_1 + \cdots + b_{n-1,i} f_{n-1}) = b_{1,i} f_1 + \cdots + b_{n-1,i} \frac{\lambda_k}{|\boldsymbol{\lambda}|} f'_{n-1}.$$

If $u_j = 0$ for $j \neq k$, then $U f_i = 0$. For the last vector we have

$$f'_{n-1} \mapsto \pi_k U \pi_k^{-1} f'_{n-1} = \frac{|\boldsymbol{\lambda}|}{\lambda_k} \pi_k U f_{n-1} = \pi_k U \frac{\pi_k \boldsymbol{\lambda}}{|\pi_k \boldsymbol{\lambda}|} - \frac{|\pi_k \boldsymbol{\lambda}|}{\lambda_k} \pi_k U e_k.$$

Since $\frac{\pi_k \boldsymbol{\lambda}}{|\pi_k \boldsymbol{\lambda}|}$ is unitary, we can write $U \frac{\pi_k \boldsymbol{\lambda}}{|\pi_k \boldsymbol{\lambda}|} = b_{1,n-1} f_1 + \cdots + b_{n-1,n-1} f_{n-1}$. Notice that $U \frac{\pi_k \boldsymbol{\lambda}}{|\pi_k \boldsymbol{\lambda}|} = 0$ if $u_j = 0$ for $j \neq k$. On the other hand $U e_k = b_{1,n} f_1 + \cdots + b_{n-1,n} f_{n-1}$, and if $u_k = 0$ then $U e_k = 0$. We collect the terms to get

$$f'_{n-1} \mapsto (b_{1,n-1} f_1 + \cdots + b_{n-1,n-1} \frac{\lambda_k}{|\boldsymbol{\lambda}|} f'_{n-1}) - \frac{|\pi_k \boldsymbol{\lambda}|}{\lambda_k} (b_{1,n} f_1 + \cdots + b_{n-1,n} \frac{\lambda_k}{|\boldsymbol{\lambda}|} f'_{n-1}).$$

Therefore, we get the matrix representation (4-6) of U_k.

The matrix $B = (b_1 \cdots b_n) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is no other than the operator U, written in terms of the orthonormal basis $f_1, \cdots, f_{n-2}, f'_{n-1}, e_k$ for $\mathbb{R}^n$, and $f_1, \cdots, f_{n-1}$ for $\mathbb{R}^{n-1}$. Since $|U| \le \frac{1}{10}$ by our hypotheses after (4-3), we get that $|B| \le \frac{1}{10}$. □

Given a standard $(n, \boldsymbol{\lambda})$-tuple $v_k + V_{k,\boldsymbol{\lambda}}$, we want to define a cap centered at $v'_k \in v_k + V_{k,\boldsymbol{\lambda}}$ as $\delta \Omega_k'^{-t} D_k^{-1} B_1$, where Ω' is the matrix

$$\Omega' = \begin{pmatrix} v_1 & \cdots & v'_k & \cdots & v_n \\ 1 & \cdots & 1 & \cdots & 1 \end{pmatrix}.$$

The next lemma is a Fourier analytic way of saying that the tubes given by $\mathcal{E}\mathbb{1}_{\alpha_k}$, for α_k in a standard $(n, \boldsymbol{\lambda})$-tuple, intersect in the ellipsoid $\delta^{-1} \Omega D_{\boldsymbol{\lambda}} B_1 \subset \mathbb{R}^n$.

Lemma 4.3. *Fix vectors $\{v_j\}_{j=1,\dots,n}$ and the corresponding standard $(n, \boldsymbol{\lambda})$-tuple $v_j + V_{j,\boldsymbol{\lambda}}$ in Definition 4.1. Let $v'_j = v_j$ for $j \neq k$ and $v'_k \in v_k + V_{k,\boldsymbol{\lambda}}$. If $\alpha_k := v'_k + \delta \Omega_k'^{-t} D_k^{-1} B_1$ and $|\boldsymbol{\lambda}| \sup_{\Omega'', j} |\Omega_j''^{-t} D_j^{-1}|^2 = \frac{1}{10}$, then for every $\delta \le 1$ we have $\tilde{\alpha}_k \subset (v'_k, -\frac{1}{2}|v'_k|^2) + \delta \Omega^{-t} D_{\boldsymbol{\lambda}}^{-1} B_1$ (recall that $\tilde{\alpha}_k$ is the lift of the set to S).*

Remark. The ellipsoid $\delta \Omega^{-t} D_{\boldsymbol{\lambda}}^{-1} B_1$ is dual to $\delta^{-1} \Omega D_{\boldsymbol{\lambda}} B_1$, which is the ellipsoid where the tubes intersect. The supremum in $|\boldsymbol{\lambda}| \sup_{\Omega'', j} |\Omega_j''^{-t} D_j^{-1}|^2$ runs over all Ω'', such that the defining vectors $\{v''_j\}$ lie in the standard $(n, \boldsymbol{\lambda})$-tuple.

This Lemma may be compared with Lemma 2.5.

Proof. We must show that

$$\tilde{\alpha}_k - \tilde{v'}_k = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \delta\Omega_k'^{-t} D_k^{-1} B_1,\ x_n = -\frac{1}{2}|x' + v_k'|^2 + \frac{1}{2}|v_k'|^2\} \subset \delta\Omega^{-t} D_{\boldsymbol{\lambda}}^{-1} B_1.$$

From (4-3) we have

$$D_{\boldsymbol{\lambda}}^{-1} W^{-1} D_{\boldsymbol{\lambda}} = I - (I+U)^{-1} U := I + \tilde{U},$$

where $|(I+U)^{-1}| \leq \frac{10}{9}$, hence $|\tilde{U}| \leq \frac{10}{9}|U| \leq \frac{1}{9}$. Therefore we get

$$\Omega'^{-t} D_{\boldsymbol{\lambda}}^{-1} = \Omega^{-t} D_{\boldsymbol{\lambda}}^{-1} (I + \tilde{U}^t),$$

and we have the expected stability result $\frac{1}{2}\Omega^{-t} D_{\boldsymbol{\lambda}}^{-1} B_1 \subset \Omega'^{-t} D_{\boldsymbol{\lambda}}^{-1} B_1 \subset 2\Omega^{-t} D_{\boldsymbol{\lambda}}^{-1} B_1$, hence it suffices to show that $\tilde{\alpha}_k \subset \delta\Omega'^{-t} D_{\boldsymbol{\lambda}}^{-1} B_1$, equivalently $\delta^{-1} D_{\boldsymbol{\lambda}} \Omega'^t \tilde{\alpha}_k \subset B_1$. To see this, we write

$$\Omega'^t x = \begin{pmatrix} v_{1,1}' x_1 + \cdots + v_{n-1,1}' x_{n-1} + x_n \\ \vdots \\ v_{1,n}' x_1 + \cdots + v_{n-1,n}' x_{n-1} + x_n \end{pmatrix} = \iota_k \Omega_k'^t x' + (x_n + \langle x', v_k' \rangle)(1, \ldots, 1).$$

We apply this to $x = (\delta\Omega_k'^{-t} D_k^{-1} z, -\frac{1}{2}|x' + v_k'|^2 + \frac{1}{2}|v_k'|^2) \in \tilde{\alpha}_k - \tilde{v}_k'$, for $z \in B_1$, to get

$$\delta^{-1} D_{\boldsymbol{\lambda}} \Omega'^t x = \iota_k z - \frac{\delta^{-1}}{2}|x'|^2 (\lambda_1, \cdots, \lambda_n).$$

We must control the norm of the above vector, but the vector z already lies in B_1, so we are left with $\frac{\delta^{-1}}{2}|x'|^2(\lambda_1, \cdots, \lambda_n)$.

We conclude by the hypotheses that

$$\left|\frac{\delta^{-1}}{2}|x'|^2(\lambda_1, \cdots, \lambda_n)\right| \leq \frac{\delta|\boldsymbol{\lambda}|}{2} |\Omega_k'^{-t} D_k^{-1} z|^2 \leq \frac{\delta}{2} |\boldsymbol{\lambda}| \sup_{\Omega'', j} |\Omega_j''^{-t} D_j^{-1}|^2 \leq \frac{1}{2}.$$

□

The condition of normalization $|\boldsymbol{\lambda}| \sup_{\Omega', j} |\Omega_j'^{-t} D_j^{-1}|^2 = \frac{1}{10}$ seems quite a restrictive one, however given any $D_{\boldsymbol{\lambda}}$ with $\beta = |\boldsymbol{\lambda}| \sup_{\Omega', j} |\Omega_j'^{-t} D_j^{-1}|^2$, we can modify it by a factor $D_{r\boldsymbol{\lambda}}$, for some $r > 0$, to get $|r\boldsymbol{\lambda}| \sup_{\Omega', j} |\Omega_k'^{-t} D_j^{-1}|^2 = r^{-1}\beta = 1$, where D_j is now defined using $r\boldsymbol{\lambda}$. The standard $(n, \boldsymbol{\lambda})$-tuple remains unaltered after normalization, as can be seen from Definition 4.1, because after multiplying $\boldsymbol{\lambda}$ by a factor r, it cancels out in the term $\frac{1}{\lambda_k} D_{k^*}$. The condition $|\boldsymbol{\lambda}| \sup_{\Omega', j} |\Omega_j'^{-t} D_j^{-1}|^2 = \frac{1}{10}$ implies that every cap lies inside a ball $\frac{1}{10} B_{|\boldsymbol{\lambda}|^{-1/2}}$.

Definition 4.4. A dilatation $D_{\boldsymbol{\lambda}}$ is *normalized* if $|\boldsymbol{\lambda}| \sup_{\Omega', k} |\Omega_k'^{-t} D_k^{-1}|^2 = \frac{1}{10}$.

Unless otherwise stated, we will assume that $D_{\boldsymbol{\lambda}}$ is normalized. Since $D_{\boldsymbol{\lambda}}$ is determined by $n-1$ parameters, we got a $(n-1)$-parameter family of standard $(n, \boldsymbol{\lambda})$-tuples around the points $\{v_j\}_{j=1,\ldots,n}$.

Definition 4.5. Suppose we are given n vectors $\{v_j\}_{j=1,\dots,n}$ and a corresponding standard $(n,\boldsymbol{\lambda})$-tuple in Definition 4.1. A δ-cap centered at $v'_k \in v_k + V_{k,\boldsymbol{\lambda}}$, for $\delta \leq 1$, is the set

$$\alpha_k := v'_k + \delta \Omega'^{-t}_k D_k^{-1} B_1, \tag{4-7}$$

where Ω' is defined by the vectors $\{v'_j\}$ and $v'_j = v_j$ for $j \neq k$.

Although caps may have large variations inside the standard $(n,\boldsymbol{\lambda})$-tuple, we have at least stability in their volume.

Lemma 4.6. *If $\alpha_k = v_k + \delta \Omega_k^{-t} D_k^{-1} B_1$ and $v'_k \in v_k + V_{k,\boldsymbol{\lambda}}$ is a vector inside the standard $(n,\boldsymbol{\lambda})$-tuple, then for $\alpha'_k = v'_k + \delta \Omega'^{-t}_k D_k^{-1} B_1$ we have $|\alpha'_k| \sim |\alpha_k|$.*

Proof. Since all the vectors v_j, for $j \neq k$, are kept fixed, Theorem 4.2 implies

$$\Omega'_k D_k = \Omega_k D_k (I + U_k),$$

where

$$|\det(I+U_k)| = \left|\det \begin{pmatrix} 1 & 0 & \cdots & -b_{1,n}\frac{|\pi_k\boldsymbol{\lambda}|}{\lambda_k} \\ 0 & 1 & \cdots & -b_{2,n}\frac{|\pi_k\boldsymbol{\lambda}|}{\lambda_k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 - b_{n-1,n}\frac{|\pi_k\boldsymbol{\lambda}|}{|\boldsymbol{\lambda}|} \end{pmatrix}\right| = \left|1 - b_{n-1,n}\frac{|\pi_k\boldsymbol{\lambda}|}{|\boldsymbol{\lambda}|}\right| \sim 1.$$

Finally $|\alpha'_k|^{-1} = |\det \Omega'_k D_k| \sim |\alpha_k|^{-1}$. □

To decompose a standard $(n,\boldsymbol{\lambda})$-tuple into caps, we need to be sure that they are at least stable at scale $B_{|\boldsymbol{\lambda}|^{-1/2}}$; otherwise, adjacent caps would be quite different between each other, and we hardly could divide the standard $(n,\boldsymbol{\lambda})$-tuple into caps. The next Lemma asserts that the caps are stable at scale $|\boldsymbol{\lambda}|^{-1/2}$, hence we can decompose safely into caps.

Lemma 4.7. *If $v'_k, v''_k \in v_k + V_{k,\boldsymbol{\lambda}}$ are vectors in a standard $(n,\boldsymbol{\lambda})$-tuple and $|v'_k - v''_k| \leq |\boldsymbol{\lambda}|^{-1/2}$, then $\frac{1}{2}\delta\Omega'^{-t}_k D_k^{-1} B_1 \subset \delta\Omega''^{-t}_k D_k^{-1} B_1 \subset 2\delta\Omega'^{-t}_k D_k^{-1} B_1$.*

Proof. Let us write $Q' = \Omega'^{-t}_k D_k^{-1}$ and $Q'' = \Omega''^{-t}_k D_k^{-1}$. It is clear that

$$\Omega''_k = \Omega'_k + (\Delta v'_k \cdots \Delta v'_k),$$

where $\Delta v'_k = v'_k - v''_k$. Hence

$$Q'^t Q''^{-t} = D_k^{-1}\Omega'^{-1}_k \Omega''_k D_k = I + (\lambda_1 D_k^{-1}\Omega'^{-1}_k \Delta v'_k \ \cdots \ \lambda_n D_k^{-1}\Omega'^{-1}_k \Delta v'_k).$$

If we write $\Delta v'_k = |\boldsymbol{\lambda}|^{-1/2} z$, for $z \in B_1$, then by $D_k^{-1}\Omega'^{-1}_k = Q'^t$ we get

$$Q'^t Q''^{-t} = I + \frac{1}{|\boldsymbol{\lambda}|^{1/2}}(\lambda_1 Q'^t z \ \cdots \ \lambda_n Q'^t z) := I + A,$$

but the operator norm of $(\lambda_1 Q'^t z \ \cdots \ \lambda_n Q'^t z)$ is $|\boldsymbol{\lambda}||Q'^t z| \leq |\boldsymbol{\lambda}||Q'|$. By our normalization of $D_{\boldsymbol{\lambda}}$ we have

$$Q''^{-t} = Q'^{-t}(I + A), \qquad \text{for } |A| \leq |\boldsymbol{\lambda}|^{1/2}|Q'| \leq \frac{1}{\sqrt{10}}.$$

Therefore $I+A$ is loosely equal to the identity and likewise $(I+A)^{-t}$. Since $Q'' = Q'(I+A)^{-t}$, we can conclude the lemma. □

4.1.4 The paraboloid in $\mathbb{R}^3$

So far our discussion has been somewhat abstract, so let us be more specific and deal with $\mathbb{R}^3$. We will show how one can recover the triplets and decomposition into caps used by Ramos as a particular instance of standard $(3, \boldsymbol{\lambda})$-tuples. As we said after Definition 4.4, we have constructed a $(n-1)$-parameter family of standard $(n, \boldsymbol{\lambda})$-tuples around any n non-coplanar points $\{v_j\}_{j=1}^n$, hence for every non-collinear triplet of points $\{v_k\}_{k=1,2,3}$ in $\mathbb{R}^2$ there is a 2-parameter family of standard $(3, \boldsymbol{\lambda})$-tuples. Ramos' triplets are, for every triplet of points, only one member of this family

Every triplet of vectors $\{v_k\}_{k=1,2,3}$ can be transformed, after translation, rotation and homogeneous dilatation, into

$$\Omega = \begin{pmatrix} 0 & 2^{-j} & w_1 \\ 0 & 0 & w_2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \text{for } |w| = 1.$$

To simplify, let us assume that $w_1 \geq 0$ and that $w_2 = \sin(2\pi\phi)$, for $|\phi| = 2^{-t}$.

We will show that Ramos' triplet around $\{v_k\}_{k=1,2,3}$ is loosely the same $(3, \boldsymbol{\lambda})$-tuple for the non-normalized dilatation

$$D_{\boldsymbol{\lambda}} = \operatorname{diag}(2^j\ 2^j\ 1).$$

We calculate first the form of a non-normalized cap $Q_k B_1 := \Omega_k^{-t} D_k^{-1} B_1$; since the set of points in the cap is $y = Q_k x$, for $|x| \leq 1$, we have that $\langle Q_k^{-t} Q_k^{-1} y, y \rangle \leq 1$. Therefore, if $\mu_{i,k}^{-2} > 0$ and $u_{i,k}$ are the eigenvalues and normalized eigenvectors of $Q_k^{-t} Q_k^{-1} = \Omega_k D_k^2 \Omega_k^t$, respectively, then $Q_k B_1$ is the ellipsoid with principal axes $u_{i,k}$ and diameters $\mu_{i,k}$. For $k = 1$ we get

$$\mu_{1,1} = \frac{1}{\sqrt{1+w_1}} \sim 1 \qquad\qquad u_{1,1} = \frac{1}{\sqrt{2}}\Big(\sqrt{1+w_1}, \frac{w_2}{\sqrt{1+w_1}}\Big)$$

$$\mu_{2,1} = \frac{\sqrt{1+w_1}}{|w_2|} \sim 2^t \qquad\qquad u_{2,1} = \frac{1}{\sqrt{2}}\Big(-\frac{w_2}{\sqrt{1+w_1}}, \sqrt{1+w_1}\Big).$$

The non-normalized cap $Q_1 B_1$ just obtained is similar to the rectangle $[-1,1] \times [-2^t, 2^t]$. The same holds for $Q_2 B_1$.

Since $2^j = \sup_k \lambda_k$ and $Q_k^{-t} = \Omega_k D_k$, we can already calculate the standard $(3, \boldsymbol{\lambda})$-tuple. The ellipsoid $\Omega_1 D_1 B_1$ is dual to the cap $\Omega_1^{-t} D_1^{-1} B_1 \approx [-1,1] \times [-2^t, 2^t]$, hence $\Omega_1 D_1 B_1 \approx [-1,1] \times [-2^{-t}, 2^{-t}]$ and so the standard triplet is

$$V_{1,\boldsymbol{\lambda}} \approx \frac{1}{30}[-2^{-j}, 2^{-j}] \times [-2^{-(j+t)}, 2^{-(j+t)}],$$

$$(0, 2^{-j}) + V_{2,\boldsymbol{\lambda}} \approx (0, 2^{-j}) + \frac{1}{30}[-2^{-j}, 2^{-j}] \times [-2^{-(j+t)}, 2^{-(j+t)}] \ \text{ and}$$

$$w + V_{3,\boldsymbol{\lambda}} \approx w + \frac{1}{30}[-1, 1] \times [-2^{-t}, 2^{-t}].$$

This triplet does not depend on the normalization of $D_{\boldsymbol{\lambda}}$; see the discussion before Definition 4.4.

The caps for $k = 3$ are calculated in a similar way to get

$$\mu_{1,3} \sim 2^{-j} \qquad u_{1,3} \approx (w_1, w_2)$$
$$\mu_{2,3} \sim 2^{t} \qquad u_{2,3} \approx (-w_2, w_1).$$

The cap $Q_3 B_1$ can have large variations in angle inside $w + V_{3,\boldsymbol{\lambda}}$, however the shape is essentially preserved.

Since $|\Omega_k^{-t} D_k^{-1}|$ equals the largest eigenvalue $\mu_{i,k}$, we can estimate $|\boldsymbol{\lambda}| \sup_{\Omega',k} |\Omega_k^{-t} D_k^{-1}|^2 \sim 2^{j+2t}$, hence multiplying the original dilatation $D_{\boldsymbol{\lambda}}$ by a factor $\sim 2^{-(j+2t)}$, we get the normalized dilatation

$$D_{\boldsymbol{\lambda}} := C\mathrm{diag}(2^{2(j+t)}\ 2^{2(j+t)}\ 2^{j+2t}), \qquad \text{for } C \sim 1,$$

and the caps α_k are, for $v_k' \in v_k + V_{k,\boldsymbol{\lambda}}$ and $C \sim 1$,

$$\begin{aligned} v_1' + \alpha_1 &\approx v_1' + C[-2^{-(j+2t)}, 2^{-(j+2t)}] \times [-2^{-(j+t)}, 2^{-(j+t)}], \\ v_2' + \alpha_2 &\approx v_2' + C[-2^{-(j+2t)}, 2^{-(j+2t)}] \times [-2^{-(j+t)}, 2^{-(j+t)}] \quad \text{and} \\ v_3' + \alpha_3 &\approx v_3' + Q_3^{-t} B_1, \end{aligned}$$

where $Q_3^{-t} B_1$ is roughly the ellipsoid with principal axes v_3' and $v_3'^{\perp}$, and diameters $2^{-2(j+t)}$ and $2^{-(j+t)}$ respectively.

The standard $(3, \boldsymbol{\lambda})$-tuple and caps we have obtained are, after dilatation if necessary, essentially the same used by Ramos [60]. This standard triplet enjoys additionally very good properties of "orthogonality", and Ramos was able to cover efficiently an arbitrary triplet of transversal sets with only these standard triplets of sets.

For the paraboloid in $\mathbb{R}^3$, it seems quite possible to cover an arbitrary triplet of transversal sets with only "one" standard $(3, \boldsymbol{\lambda})$-tuple given by the non-normalized dilatation

$$D_{\boldsymbol{\lambda}} = \mathrm{diag}(\rho\ \rho\ 1), \qquad \text{for some } \rho \geq 1.$$

These triplets are more flexible than those used by Ramos; unfortunately, these standard triplets do not enjoy, in general, the very good properties of orthogonality that do have the triplets used by Ramos; however, I think that they still enjoy some weak orthogonality, which may suffices to prove Theorem 1.8. I do not pursue this matter here.

Ramos' work shows that only a specific instance of standard $(n, \boldsymbol{\lambda})$-tuple suffices to get the sharp dependence on transversality, and this brings hope that the full problem may be solved using Ramos' strategy.

4.2 Induction on Scales

In this section we show that the multiscale analysis is efficient, when changing scales, over standard $(n, \boldsymbol{\lambda})$-tuples. We will make the induction over "optimal" parallelepipeds $\delta^{-1}P^* =$

$\delta^{-1}\Omega D_{\boldsymbol{\lambda}}$, where $D_{\boldsymbol{\lambda}}$ is a normalized dilatation. Since the arguments were already described in Section 2.3, we only sketch the main steps.

Fix a standard $(n,\boldsymbol{\lambda})$-tuple of sets $\{U_k\}_{k=1,\dots,n}$ and let $K(\delta^{-1})$ be the best constant that satisfies, for every n-tuple of functions $\{f_k\}$ supported in the standard $(n,\boldsymbol{\lambda})$-tuple, the inequality

$$\int_{\delta^{-1}P^*} |\prod_{k=1}^{n} \mathcal{E} f_k|^{\frac{2}{n-1}}\, dx \le K(\delta^{-1}) \prod_{k=1}^{n} \|f_k\|_2^{\frac{2}{n-1}}, \qquad \text{for } \delta \le 1.$$

We aim to prove the next theorem.

Theorem 4.8. *For every standard $(n,\boldsymbol{\lambda})$-tuple of sets it holds that*

$$K(R) \le C_0 K(R^{\frac{1}{2}}), \tag{4-8}$$

where C_0 does not depend on R or θ.

We divide each U_k into δ-caps α_k, and recall that $\mathcal{E} f_{k,\alpha_k} = \mathcal{E}(f_k \mathbb{1}_{\alpha_k})$. Recall our heuristic approximation

$$\mathcal{E} f_k(x) \approx \sum_{\alpha_k} \mathcal{E} f_{k,\alpha_k}(z) e(-\langle x'-z', c_{\alpha_k}\rangle + \frac{1}{2}(x_n - z_n)|c_{\alpha_k}|^2), \qquad \text{for } x \in z + \delta^{-1}P^*,$$

which was formalized by the reproducing formula

$$\mathcal{E} f_{k,\alpha_k} = \frac{1}{|\alpha_k|} \mathcal{E} h_{k,\alpha_k} * \big(\frac{1}{|\delta^{-1}P^*|} \zeta_{\delta^{-1}P^*} \mathcal{E} \mathbb{1}_{\alpha_k}\big), \tag{4-9}$$

where h_{k,α_k} is essentially equal to f_{k,α_k}. Using (4-9) we get

$$\begin{aligned} \int_{z+\delta^{-1}P^*} |\prod_{k=1}^{n} \mathcal{E} f_k|^{\frac{2}{n-1}}\, dx &= \int_{z+\delta^{-1}P^*} |\sum_{\alpha_1,\cdots,\alpha_n} \prod_{k=1}^{n} \mathcal{E} f_{k,\alpha_k}\Big|^{\frac{2}{n-1}}\, dx, \\ &= \int_{z+\delta^{-1}P^*} \Big| \int \prod_k \frac{1}{|\delta^{-1}P^*|} \zeta_{\delta^{-1}P^*}(x-y_k) \\ &\qquad \prod_k \sum_{\alpha_k} \frac{1}{|\alpha_k|} \mathcal{E}\mathbb{1}_{\alpha_k}(x-y_k) \mathcal{E} h_{k,\alpha_k}(y_k)\, d\boldsymbol{y}\Big|^{\frac{2}{n-1}}\, dx. \end{aligned} \tag{4-10}$$

If $\frac{2}{n-1}$ were greater than or equal to one, which in fact happens for $n=3$, we could use the Minkowski integral inequality to exchange the variables of integration, but unfortunately $\frac{2}{n-1} < 1$ most of the time, so it is not now so simple as in Chapter 2.

Since each function $\mathcal{E}\mathbb{1}_{\alpha_k}(x-y_k)\mathcal{E} h_{k,\alpha_k}(y_k)$ is essentially constant with respect to y_k on parallelepipeds $\delta^{-1}P^*$, we can factor these terms from the inner integral. In fact, letting

x fixed, the functions $(\mathcal{E}\mathbb{1}_{\alpha_k}(x-\cdot)\mathcal{E}h_{k,\alpha_k})^\vee = (\mathbb{1}_{\alpha_k}e(\langle x',\cdot\rangle - \frac{1}{2}x_n|\cdot|^2)d\sigma(-\cdot) * h_{k,\alpha_k}d\sigma$ are supported in δP, so we have a reproducing formula

$$\begin{aligned} H_x(y_k) &:= \frac{1}{|\delta^{-1}P^*|}\zeta_{\delta^{-1}P^*}(x-y_k)\sum_{\alpha_k}\mathcal{E}\mathbb{1}_{\alpha_k}(x-y_k)\mathcal{E}h_{k,\alpha_k}(y_k) \\ &= \Big(\frac{1}{|\delta^{-1}P^*|}\zeta_{\delta^{-1}P^*}(x-y_k)\sum_{\alpha_k}\mathcal{E}\mathbb{1}_{\alpha_k}(x-\cdot)\mathcal{E}h_{k,\alpha_k}\Big) * \hat{\zeta}_{\delta P}(y_k). \end{aligned}$$

We need now the following lemma; see [50, Appendix].

Lemma 4.9. *If H is a function whose Fourier transform is supported in a parallelepiped R centered at the origin, then for $q<1$ we have*

$$\Big(\int |H(y)|\,dy\Big)^q \lesssim |R^*|^{q-1}\int |H(y)|^q\,dy, \tag{4-11}$$

where R^* is the dual parallelepiped.

Proof. Writing $H = H * \hat{\zeta}_R$ and using Young's inequality for convolutions we have

$$\begin{aligned} \int |H(y)|\,dy &\le \|H\|_\infty^{1-q}\int |H(y)|^q\,dy \\ &\lesssim |R^*|^{q-1}\|H\|_1^{1-q}\int |H(y)|^q\,dy, \end{aligned}$$

where we used $|R|^{-1} = |R^*|$. We divide both sides by $\|H\|_1^{1-q}$ to get

$$\Big(\int |H(y)|\,dy\Big)^q \lesssim |R^*|^{q-1}\int |H(y)|^q\,dy.$$

□

We apply this lemma, together with Fubini and the change of variables $x \mapsto x+z$ and $y_k \mapsto y_k + z$, to continue (4-10) as

$$\begin{aligned} \int_{z+\delta^{-1}P^*} |\prod_{k=1}^n \mathcal{E}f_k|^{\frac{2}{n-1}}\,dx \lesssim \iint_{\delta^{-1}P^*} &\prod_k \frac{1}{|\delta^{-1}P^*|}|\zeta_{\delta^{-1}P^*}(x-y_k)|^{\frac{2}{n-1}} \\ &\Big|\prod_k\sum_{\alpha_k}\frac{1}{|\alpha_k|}\mathcal{E}\mathbb{1}_{\alpha_k}(x-y_k)\mathcal{E}h_{k,\alpha_k}(y_k+z)\Big|^{\frac{2}{n-1}}\,dxd\boldsymbol{y}. \end{aligned} \tag{4-12}$$

To eliminate the dependence on y_k in the inner integral, we notice that

$$\begin{aligned} \mathbb{1}_{\delta^{-1}P^*}(x)\prod_k \frac{1}{|\delta^{-1}P^*|}|\zeta_{\delta^{-1}P^*}(x-y_k)|^{\frac{2}{n-1}} &\le \mathbb{1}_{\delta^{-1}P^*}(x)\prod_k \frac{1}{|\delta^{-1}P^*|}\sup_{x'\in\delta^{-1}P^*}|\zeta_{\delta^{-1}P^*}(x'-y_k)|^{\frac{2}{n-1}} \\ &= \mathbb{1}_{\delta^{-1}P^*}(x)\prod_k \eta_{\delta^{-1}P^*}(y_k), \end{aligned}$$

where $\eta_{\delta^{-1}P^*}(y) := \frac{1}{|\delta^{-1}P^*|}\sup_{x'\in\delta^{-1}P^*}|\zeta_{\delta^{-1}P^*}(x'-y)|^{\frac{2}{n-1}}$ and $\int\eta_{\delta^{-1}P^*}\lesssim 1$. Therefore, we can replace (4-12) by

$$\int_{z+\delta^{-1}P^*}|\prod_{k=1}^{n}\mathcal{E}f_k|^{\frac{2}{n-1}}\,dx \lesssim \int\prod_k \eta_{\delta^{-1}P^*}(y_k)$$
$$\Big[\int_{\delta^{-1}P^*}|\prod_k\sum_{\alpha_k}\frac{1}{|\alpha_k|}\mathcal{E}\mathbb{1}_{\alpha_k}(x-y_k)\mathcal{E}h_{k,\alpha_k}(z+y_k)|^{\frac{2}{n-1}}\,dx\Big]\,d\boldsymbol{y}. \quad (4\text{-}13)$$

Now, to simplify further, we define the functions

$$g_k(\xi)=\sum_{\alpha_k}\frac{1}{|\alpha_k|}e(\langle y_k',\xi\rangle+\frac{1}{2}y_{k,n}\varphi(\xi))\mathcal{E}h_{k,\alpha_k}(z+y_k)\mathbb{1}_{\alpha_k}(\xi),$$

to rewrite (4-13) as

$$\int_{z+\delta^{-1}P^*}|\prod_{k=1}^{n}\mathcal{E}f_k|^{\frac{2}{n-1}}\,dx \lesssim \int\prod_k\eta_{\delta^{-1}P^*}(y_k)\Big[\int_{\delta^{-1}P^*}|\prod_k\mathcal{E}g_k|^{\frac{2}{n-1}}\,dx\Big]\,d\boldsymbol{y}.$$

We apply the multilinear estimate to g_k to obtain

$$\begin{aligned}\int_{z+\delta^{-1}P^*}|\prod_{k=1}^{n}\mathcal{E}f_k|^{\frac{2}{n-1}}\,dx &\le K(\delta^{-1})\int\prod_k\eta_{\delta^{-1}P^*}(y_k)\prod_k\|g_k\|_2^{\frac{2}{n-1}}\,d\boldsymbol{y}\\ &= K(\delta^{-1})\int\prod_k\eta_{\delta^{-1}P^*}(y_k)\\ &\qquad\Big[\prod_k|\alpha_k|^{-1/2}\big(\sum_{\alpha_k}|\mathcal{E}h_{k,\alpha_k}(z+y_k)|^2\big)^{1/2}\Big]^{\frac{2}{n-1}}\,d\boldsymbol{y}.\end{aligned}$$

We can manipulate $|\alpha_k|$ as a constant because of Lemma 4.6. Integrating both sides over $\frac{1}{|\delta^{-1}P^*|}\int_{\delta^{-2}P^*}dz$ we get

$$\int_{\delta^{-2}P^*}|\prod_{k=1}^{n}\mathcal{E}f_k|^{\frac{2}{n-1}}\,dx \lesssim \frac{K(\delta^{-1})}{|\delta^{-1}P^*|\prod_k|\alpha_k|^{1/(n-1)}}\int\prod_k\eta_{\delta^{-1}P^*}(y_k)$$
$$\Big[\int_{\delta^{-2}P^*}\prod_k\big(\sum_{\alpha_k}|\mathcal{E}h_{k,\alpha_k}(z+y_k)|^2\big)^{1/(n-1)}\,dz\Big]\,d\boldsymbol{y}, \quad (4\text{-}14)$$

where the term $|\delta^{-1}P^*|\prod_k|\alpha_k|^{1/(n-1)}\sim\theta^{-1/(n-1)}$. Since the Fourier transform converts translations into modulations, it suffices to establish a bound of the square function

$$\int_{\delta^{-2}P^*}\prod_k\big(\sum_{\alpha_k}|\mathcal{E}h_{k,\alpha_k}(z)|^2\big)^{1/(n-1)}\,dz$$

by some expression involving $\prod_k\|h_k\|_2$.

Recall that the functions h_{k,α_k} are supported in caps $\alpha_k = c_{\alpha_k} + \delta\Omega_k'^{-t} D_k^{-1} B_1$ (see Definition 4.5), and $\alpha_k \subset c_{\alpha_k} + B_{\delta|\boldsymbol{\lambda}|^{-1/2}}$ by our normalization condition in Definition 4.4. Hence $|\mathcal{E}h_{k,\alpha_k}|^2$ is supported in a parallelepiped of dimensions $\delta|\boldsymbol{\lambda}|^{-1/2} \times \cdots \delta|\boldsymbol{\lambda}|^{-1/2} \times \delta^2|\boldsymbol{\lambda}|^{-1}$ and then we get the reproducing formula $|\mathcal{E}h_{k,\alpha_k}|^2 = \frac{1}{|T_{\alpha_k}|}|\mathcal{E}h_{k,\alpha_k}|^2 * \zeta_{T_{\alpha_k}}$, where $\zeta_{T_{\alpha_k}}$ is essentially supported in a tube T_{α_k} of dimensions $\delta^{-1}|\boldsymbol{\lambda}|^{1/2} \times \cdots \times \delta^{-1}|\boldsymbol{\lambda}|^{1/2} \times \delta^{-2}|\boldsymbol{\lambda}|$ and $|T_{\alpha_k}| := \int \zeta_{T_{\alpha_k}} \sim \delta^{-(n+1)}|\boldsymbol{\lambda}|^{\frac{n+1}{2}}$. Now, we want to use the adjusted version (2-31) of the multilinear Kakeya estimate, Theorem 1.5.

We estimate the square function inside brackets in (4-14), replacing $\zeta_{T_{\alpha_k}}$ by $\mathbb{1}_{T_{\alpha_k}}$ in the reproducing formula of $|\mathcal{E}h_{k,\alpha_k}|^2$. Since $P^* \subset \mathbb{R}^{n-1} \times I_{|\boldsymbol{\lambda}|}$ for $I_{|\boldsymbol{\lambda}|} = [-|\boldsymbol{\lambda}|, |\boldsymbol{\lambda}|]$, it is not hard to see that

$$(|\mathcal{E}h_{k,\alpha_k}|^2 * \mathbb{1}_{T_{\alpha_k}})\mathbb{1}_{\delta^{-2}P^*} \leq (|\mathcal{E}h_{k,\alpha_k}|^2 \mathbb{1}_{\mathbb{R}^{n-1}\times I_{2\delta^{-2}|\boldsymbol{\lambda}|}}) * \mathbb{1}_{T_{\alpha_k}},$$

hence we can apply the multilinear estimate (2-31) to the square function inside brackets in (4-14), replacing μ_{α_k} by $\frac{1}{|T_{\alpha_k}|}|\mathcal{E}h_{k,\alpha_k}|^2\mathbb{1}_{\mathbb{R}^{n-1}\times I_{2\delta^{-2}|\boldsymbol{\lambda}|}}$, to get

$$\int_{\delta^{-2}P^*} \prod_k \Big(\sum_{\alpha_k} |\mathcal{E}h_{k,\alpha_k}(z)|^2\Big)^{1/(n-1)} dz \lesssim$$
$$\delta^{\frac{2n}{n-1}}|\boldsymbol{\lambda}|^{-\frac{n}{n-1}}\theta^{-\frac{1}{n-1}} \prod_k \Big(\sum_{\alpha_k} \|\mathcal{E}h_{k,\alpha_k}\|^2_{L^2(\mathbb{R}^{n-1}\times I_{2\delta^{-2}|\boldsymbol{\lambda}|})}\Big)^{\frac{1}{n-1}}.$$

Using Lemma 1.1 we have $\|\mathcal{E}f\|_{L^2(\mathbb{R}^{n-1}\times I_{2\delta^{-2}|\boldsymbol{\lambda}|})} \lesssim \delta^{-1}|\boldsymbol{\lambda}|^{1/2}\|f\|_2$, so we reach our desired estimate of the square function

$$\int_{\delta^{-2}P^*} \prod_k \Big[\sum_{\alpha_k} |\mathcal{E}h_{k,\alpha_k}(z)|^2\Big]^{1/(n-1)} dz \lesssim \theta^{-\frac{1}{n-1}} \prod_k \|h_k\|_{L^2}^{\frac{2}{n-1}}.$$

Plugging this in (4-14) we get

$$\begin{aligned}\int_{\delta^{-2}P^*} \Big|\prod_{k=1}^n \mathcal{E}f_k\Big|^{\frac{2}{n-1}} dx &\lesssim K(\delta^{-1})\Big(\int \eta_{\delta^{-1}P^*}\Big)^n \prod_k \|h_k\|_{L^2} \\ &\lesssim K(\delta^{-1}) \prod_k \|f_k\|_{L^2},\end{aligned}$$

which is what we wanted.

Once we have proved the inductive bound (4-8), we iterate it to reach

$$K(R) \leq C_0^N K(R^{1/2^N}).$$

If we choose $N = \lfloor \log_2 \log_2 R \rfloor + 1$, then we have

$$K(R) \leq C(\log R)^C K(2).$$

Hence, to prove the sharp multilinear inequality for standard $(n, \boldsymbol{\lambda})$-tuples, it suffices to prove that $K(2) \lesssim \theta^{-\frac{1}{n-1}}$. The standard $(n, \boldsymbol{\lambda})$-tuples seem to be rather general to allow for a proof of this fact, however I expect that in a narrower selection of $(n, \boldsymbol{\lambda})$-tuples this bound could be proved, as in $\mathbb{R}^3$.

Bibliography

[1] Bennett, J.: Aspects of multilinear harmonic analysis related to transversality. En: *Harmonic analysis and partial differential equations* Vol. 612. Amer. Math. Soc., Providence, RI, 2014, p. 1–28

[2] Bennett, J. ; Carbery, A. ; Tao, T.: On the multilinear restriction and Kakeya conjectures. En: *Acta Math.* 196 (2006), Nr. 2, p. 261–302

[3] Bourgain, J.: Besicovitch type maximal operators and applications to Fourier analysis. En: *Geom. Funct. Anal.* 1 (1991), Nr. 2, p. 147–187

[4] Bourgain, J.: On the restriction and multiplier problems in $\mathbf{R}^3$. En: *Geometric aspects of functional analysis (1989–90)* Vol. 1469. Springer, Berlin, 1991, p. 179–191

[5] Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. En: *Geom. Funct. Anal.* 3 (1993), Nr. 2, p. 107–156

[6] Bourgain, J.: Hausdorff dimension and distance sets. En: *Israel J. Math.* 87 (1994), Nr. 1-3, p. 193–201

[7] Bourgain, J.: Estimates for cone multipliers. En: *Geometric aspects of functional analysis (Israel, 1992–1994)* Vol. 77. Birkhäuser, Basel, 1995, p. 41–60

[8] Bourgain, J.: Some new estimates on oscillatory integrals. En: *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)* Vol. 42. Princeton Univ. Press, Princeton, NJ, 1995, p. 83–112

[9] Bourgain, J.: On the dimension of Kakeya sets and related maximal inequalities. En: *Geom. Funct. Anal.* 9 (1999), Nr. 2, p. 256–282

[10] Bourgain, J. ; Demeter, C. *Decouplings for curves and hypersurfaces with nonzero Gaussian curvature.* arXiv:1409.1634 [math.CA]

[11] Bourgain, J. ; Demeter, C.: The proof of the l^2 decoupling conjecture. En: *Ann. of Math. (2)* 182 (2015), Nr. 1, p. 351–389

[12] Bourgain, J. ; Demeter, C.: Decouplings for surfaces in $\mathbb{R}^4$. En: *J. Funct. Anal.* 270 (2016), Nr. 4, p. 1299–1318

[13] Bourgain, J. ; Demeter, C.: Mean value estimates for Weyl sums in two dimensions. En: *J. Lond. Math. Soc. (2)* 94 (2016), Nr. 3, p. 814–838

[14] Bourgain, J. ; Demeter, C.: A study guide for the l^2 decoupling theorem. En: *Chin. Ann. Math. Ser. B* 38 (2017), Nr. 1, p. 173–200

[15] Bourgain, J. ; Demeter, C. ; Guth, L.: Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three. En: *Ann. of Math. (2)* 184 (2016), Nr. 2, p. 633–682

[16] Bourgain, J. ; Guth, L.: Bounds on oscillatory integral operators based on multilinear estimates. En: *Geom. Funct. Anal.* 21 (2011), Nr. 6, p. 1239–1295

[17] Carbery, A. ; Valdimarsson, S. I.: The endpoint multilinear Kakeya theorem via the Borsuk-Ulam theorem. En: *J. Funct. Anal.* 264 (2013), Nr. 7, p. 1643–1663

[18] Carleson, L. ; Sjölin, P.: Oscillatory integrals and a multiplier problem for the disc. En: *Studia Math.* 44 (1972), p. 287–299. (errata insert). – Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, III

[19] Cho, C. ; Lee, J.: Improved restriction estimate for hyperbolic surfaces in $\mathbb{R}^3$. En: *J. Funct. Anal.* 273 (2017), Nr. 3, p. 917–945

[20] Córdoba, A.: The Kakeya maximal function and the spherical summation multipliers. En: *Amer. J. Math.* 99 (1977), Nr. 1, p. 1–22

[21] Drury, S. W.: L^p estimates for the X-ray transform. En: *Illinois J. Math.* 27 (1983), Nr. 1, p. 125–129

[22] Dvir, Z.: On the size of Kakeya sets in finite fields. En: *J. Amer. Math. Soc.* 22 (2009), Nr. 4, p. 1093–1097

[23] Elekes, G. ; Sharir, M.: Incidences in three dimensions and distinct distances in the plane. En: *Combin. Probab. Comput.* 20 (2011), Nr. 4, p. 571–608

[24] Erdoğan, M. B.: A bilinear Fourier extension theorem and applications to the distance set problem. En: *Int. Math. Res. Not.* (2005), Nr. 23, p. 1411–1425

[25] Erdoğan, M. B.: On Falconer's distance set conjecture. En: *Rev. Mat. Iberoam.* 22 (2006), Nr. 2, p. 649–662

[26] Erdös, P.: On sets of distances of n points. En: *Amer. Math. Monthly* 53 (1946), p. 248–250

[27] Falconer, K. J.: On the Hausdorff dimensions of distance sets. En: *Mathematika* 32 (1985), Nr. 2, p. 206–212 (1986)

[28] Falconer, K. J.: *Cambridge Tracts in Mathematics.* Vol. 85: *The geometry of fractal sets*. Cambridge University Press, Cambridge, 1986. – xiv+162 p.

[29] Federer, H.: *Geometric measure theory.* Springer-Verlag New York Inc., New York, 1969 (Die Grundlehren der mathematischen Wissenschaften, Band 153). – xiv+676 p.

[30] Fefferman, C.: Inequalities for strongly singular convolution operators. En: *Acta Math.* 124 (1970), p. 9–36

[31] Fefferman, C.: A note on spherical summation multipliers. En: *Israel J. Math.* 15 (1973), p. 44–52

[32] Fefferman, C.: Recent progress in classical Fourier analysis. (1975), p. 95–118

[33] Fefferman, Charles: The multiplier problem for the ball. En: *Ann. of Math. (2)* 94 (1971), p. 330–336

[34] Guth, L. *Restriction estimates using polynomial partitioning II.* arXiv:1603.04250 [math.CA]

[35] Guth, L.: The endpoint case of the Bennett-Carbery-Tao multilinear Kakeya conjecture. En: *Acta Math.* 205 (2010), Nr. 2, p. 263–286

[36] Guth, L.: A short proof of the multilinear Kakeya inequality. En: *Math. Proc. Cambridge Philos. Soc.* 158 (2015), Nr. 1, p. 147–153

[37] Guth, L.: A restriction estimate using polynomial partitioning. En: *J. Amer. Math. Soc.* 29 (2016), Nr. 2, p. 371–413

[38] Guth, L. ; Katz, N. H.: On the Erdős distinct distances problem in the plane. En: *Ann. of Math. (2)* 181 (2015), Nr. 1, p. 155–190

[39] Hörmander, L.: Oscillatory integrals and multipliers on FL^p. En: *Ark. Mat.* 11 (1973), p. 1–11

[40] Iosevich, A. ; Rudnev, M.: Distance measures for well-distributed sets. En: *Discrete Comput. Geom.* 38 (2007), Nr. 1, p. 61–80

[41] Katz, N. H. ; Ł aba, I. ; Tao, T.: An improved bound on the Minkowski dimension of Besicovitch sets in $\mathbf{R}^3$. En: *Ann. of Math. (2)* 152 (2000), Nr. 2, p. 383–446

[42] Katz, N. H. ; Tao, T.: New bounds for Kakeya problems. En: *J. Anal. Math.* 87 (2002), p. 231–263. – Dedicated to the memory of Thomas H. Wolff

[43] Katz, N. H. ; Zahl, J. *An improved bound on the Hausdorff dimension of Besicovitch sets in* $\mathbf{R}^3$. arXiv:1704.07210 [math.CA]

[44] Kaufman, R.: On Hausdorff dimension of projections. En: *Mathematika* 15 (1968), p. 153–155

[45] Kenig, C. E. ; Ponce, G. ; Vega, L.: A bilinear estimate with applications to the KdV equation. En: *J. Amer. Math. Soc.* 9 (1996), Nr. 2, p. 573–603

[46] Klainerman, S. ; Machedon, M.: Space-time estimates for null forms and the local existence theorem. En: *Comm. Pure Appl. Math.* 46 (1993), Nr. 9, p. 1221–1268

[47] Ł aba, I. ; Tao, T.: An x-ray transform estimate in $\mathbb{R}^n$. En: *Rev. Mat. Iberoamericana* 17 (2001), Nr. 2, p. 375–407

[48] Lee, S.: Bilinear restriction estimates for surfaces with curvatures of different signs. En: *Trans. Amer. Math. Soc.* 358 (2006), Nr. 8, p. 3511–3533

[49] Loomis, L. H. ; Whitney, H.: An inequality related to the isoperimetric inequality. En: *Bull. Amer. Math. Soc* 55 (1949), p. 961–962

[50] Lucà, R. ; Rogers, K.: Average decay of the Fourier transform of measures with applications. En: *arXiv:1503.00105 [math.CA]* (2015)

[51] Mattila, P.: Spherical averages of Fourier transforms of measures with finite energy; dimension of intersections and distance sets. En: *Mathematika* 34 (1987), Nr. 2, p. 207–228

[52] Mattila, P.: *Cambridge Studies in Advanced Mathematics*. Vol. 44: *Geometry of sets and measures in Euclidean spaces*. Cambridge University Press, Cambridge, 1995. – xii+343 p.. – Fractals and rectifiability

[53] Mattila, P.: Hausdorff dimension, projections, and the Fourier transform. En: *Publ. Mat.* 48 (2004), Nr. 1, p. 3–48

[54] Mattila, P.: *Cambridge Studies in Advanced Mathematics*. Vol. 150: *Fourier analysis and Hausdorff dimension*. Cambridge University Press, Cambridge, 2015. – xiv+440 p.

[55] Moyua, A. ; Vargas, A. ; Vega, L.: Schrödinger maximal function and restriction properties of the Fourier transform. En: *Internat. Math. Res. Notices* (1996), Nr. 16, p. 793–815

[56] Moyua, A. ; Vargas, A. ; Vega, L.: Restriction theorems and maximal operators related to oscillatory integrals in $\mathbf{R}^3$. En: *Duke Math. J.* 96 (1999), Nr. 3, p. 547–574

[57] Oberlin, D. M. ; Stein, E. M.: Mapping properties of the Radon transform. En: *Indiana Univ. Math. J.* 31 (1982), Nr. 5, p. 641–650

[58] Ou, Y. ; Wang, H. *A cone restriction estimate using polynomial partitioning.* arXiv:1704.05485 [math.CA]

[59] Pierce, L. *The Vinogradov Mean Value Theorem [after Wooley, and Bourgain, Demeter and Guth].* arXiv:1707.00119 [math.NT]

[60] Ramos, J.: *A trilinear restriction estimate with sharp dependence on transversality.* – To appear in American Journal of Mathematics

[61] Sjölin, P.: Estimates of spherical averages of Fourier transforms and dimensions of sets. En: *Mathematika* 40 (1993), Nr. 2, p. 322–330

[62] Stein, E. M.: *Princeton Mathematical Series.* Vol. 43: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton University Press, Princeton, NJ, 1993. – xiv+695 p.. – With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III

[63] Strichartz, R. S.: Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. En: *Duke Math. J.* 44 (1977), Nr. 3, p. 705–714

[64] Tao, T.: The Bochner-Riesz conjecture implies the restriction conjecture. En: *Duke Math. J.* 96 (1999), Nr. 2, p. 363–375

[65] Tao, T.: *Lectures on Restriction theorems and applications – Math 254B.* `https://www.math.ucla.edu/~tao/254b.1.99s`. Spring 1999. – Online; accessed 11 July 2017

[66] Tao, T.: A sharp bilinear restrictions estimate for paraboloids. En: *Geom. Funct. Anal.* 13 (2003), Nr. 6, p. 1359–1384

[67] Tao, T.: Some recent progress on the restriction conjecture. En: *Fourier analysis and convexity.* Birkhäuser Boston, Boston, MA, 2004 (Appl. Numer. Harmon. Anal.), p. 217–243

[68] Tao, T. ; Vargas, A. ; Vega, L.: A bilinear approach to the restriction and Kakeya conjectures. En: *J. Amer. Math. Soc.* 11 (1998), Nr. 4, p. 967–1000

[69] Temur, F.: A Fourier restriction estimate for surfaces of positive curvature in $\mathbb{R}^6$. En: *Rev. Mat. Iberoam.* 30 (2014), Nr. 3, p. 1015–1036

[70] Tomas, P. A.: A restriction theorem for the Fourier transform. En: *Bull. Amer. Math. Soc.* 81 (1975), p. 477–478

[71] Vargas, A.: Restriction theorems for a surface with negative curvature. En: *Math. Z.* 249 (2005), Nr. 1, p. 97–111

[72] Wolff, T.: An improved bound for Kakeya type maximal functions. En: *Rev. Mat. Iberoamericana* 11 (1995), Nr. 3, p. 651–674

[73] Wolff, T.: Decay of circular means of Fourier transforms of measures. En: *Internat. Math. Res. Notices* (1999), Nr. 10, p. 547–567

[74] Wolff, T.: Local smoothing type estimates on L^p for large p. En: *Geom. Funct. Anal.* 10 (2000), Nr. 5, p. 1237–1288

[75] Wolff, T.: A sharp bilinear cone restriction estimate. En: *Ann. of Math. (2)* 153 (2001), Nr. 3, p. 661–698

[76] Wolff, T.: *University Lecture Series*. Vol. 29: *Lectures on harmonic analysis*. American Mathematical Society, Providence, RI, 2003. – x+137 p.. – With a foreword by Charles Fefferman and preface by Izabella Łaba, Edited by Łaba and Carol Shubin

[77] Zygmund, A.: On Fourier coefficients and transforms of functions of two variables. En: *Studia Math.* 50 (1974), p. 189–201