Duo, quasi-duo and ascending chain condition principal properties over skew PBW extensions

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Dedicated to

My husband, Wilfredo.

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I thank my daughter, Karol, for being my motivation for life, to my husband for being a constant support in this journey, for being there whenever I needed it and for motivating me to be better every day. I thank my parents for their constant and wonderful advice and their unconditional love, to my father for teaching me the world of mathematics and for being my model of life. I thank my thesis director Professor Armando Reyes for his patience, advice and teachings.

Contents

Introduction

Ore extensions (introduced by Ore [33]) have been one of the most studied non-commutative structures in the last century. The skew polynomial rings (see Definition 1.1.1) are an Ore extensions that thanks to its similarity to the classical polynomial ring, currently seeks to "copy" the properties that already have the classic polynomial ring such as Noetherianity, some homological properties, the characterization of ideals, and others. In particular, and as the first topic of interest in this work, Marks [26] examined an extreme situation for skew polynomial rings: he asked when every left (or right) ideal is two-sided. It is important to say that for an ordinary polynomial ring, this case can no occur unless the ring is commutative (i.e., an ordinary polynomial ring is one-sided duo only if it is commutative), as it was proved by Hirano, Hong, Kim and Park ([13], Lemma 3). This result was extended in [25], Lemma 3.3, and precisely, Marks [26] obtained further generalizations of these results: he showed that if a non-commutative Ore extension $R[x;\sigma,\delta]$ which is a duo ring on one side exists, then it has to be right duo, σ must be non-injective and $\delta \neq 0$ ([26], Theorems 1 and 2). He also obtained a list of necessary conditions to guarantee that the Ore extension $R[x;\sigma,\delta]$ to be right duo. Nevertheless, Matczuk [28] proved that non-commutative skew polynomial ring which are right duo rings do exist and that the necessary conditions obtained by Marks are not sufficient for the skew polynomial ring to be right duo. Actually, Matczuk's paper is one of the most important articles about the characterization of non-commutative rings which are duo rings.

Given that the skew PBW extensions introduced by Gallego and Lezama [9] are a type of non-commutative ring more general than Ore extensions (of injective type, i.e., when σ is injective), it is interesting to analyze how much the results obtained by Marks and Matczuk can be generalized to these extensions. As we will see in Section 1.2, the condition that Marks and Matczuk impose is that the endomorphism σ is not injective, which has as a particular consequence that the Noetherianity of the Ore extension (and hence of the skew PBW extensions) fails. This fact is not very convenient for the study of these extensions, as we can appreciated in previous works where the Noetherianity is a key ingredient to the characterization of several ring-theoretical properties (see [23], [24], [32], and [35]-[48]). For this reason it is necessary to head towards another weakest notion, the quasi-duo rings whose definition is presented in the Section 1.3.

Matczuk [28] opens the way to studying quasi-duo rings over the Ore extension $R[x;\sigma,\delta]$. Two years later, Leroy, Matczuk and Puczyłowski [21] gave a valuable characterization of the Ore extension $R[x;\sigma,\delta]$ being quasi-duo ([21], Theorem 3.4). The same authors, but now in [22], perform the study of quasi-duo property on $\mathbb{Z}$-graded rings. With all these treatments in mind and considering a graded version of skew PBW extensions (in general these extensions are not graded, i.e., they are not trivially graded), it is our interest to consider the definition of a graded skew PBW extension introduced recently by Suárez [45], and then present examples of graded skew PBW extensions with the aim of determining the quasi-duo property in some examples of skew PBW extensions. A remarkable fact is that during the development of this work, Bien and Öinert [2] found a result on the property of being quasi-duo for Ore extensions of derivation type (see Section 1.3). Since these extensions are also examples of skew PBW extensions, we will provide more examples of skew PBW extensions that are not quasi-duo (the negative answer for this property over skew PBW extensions due to the results obtained by Bien and Öinert).

The second topic of interest of this work is about the ascending chain condition on principal right (resp. left) ideals of non-commutative rings. This topic has been studied in different papers such as Anderson [1], Grams [11], Renault [34] and Mazurek and Ziembowski [29]. More recently, Nasr-Isfahani [31] considered the ascending chain condition on principal left (ACCPL) (resp. right) ideals of the Ore extension $R[x;\sigma;\delta]$, and he gave a characterization of skew polynomial rings $R[x;\sigma;\delta]$ that are domains and satisfy the ascending chain condition on principal left (resp. right) ideals. In the same paper, the author also proved that if R is an σ-rigid ring (see Krempa [17]) that satisfies the ascending chain condition on right annihilators and ascending chain condition on principal right (resp. left) ideals, then the Ore extension $R[x;\sigma;\delta]$ and skew power series ring $R[[x;\sigma]]$ also satisfy the ascending chain condition on principal right (ACCPR) (resp. left) ideals. Similarly to the first topic of this work, we ask ourselves for the ACCPL condition for the more general context of skew PBW extensions. Fortunately, we were able to generalize the results obtained by Nasr-Isfahani to the family of skew PBW extensions. All these results are presented on Chapter 2 of this work[1].

Next we present the structure of this work: in Chapter 1, Section 1, we recall some definitions and necessary results for the entire document. In Section 2 we present the main results obtained by Marks [26] and Matczuk [28] about the notion of duo ring over the Ore extension $R[x;\sigma,\delta]$. Now, Section 3 contains the notion of quasi-duo ring and the results on Ore extensions which are $\mathbb{Z}$-graded rings (we present the treatment developed by given by Leroy, Matczuk and Puczyłowski [22]). Finally, we conclude this chapter with the most relevant conclusions of the work of Bien and Öinert [2] for the quasi-duo property of Ore extensions of derivation type. Now, Chapter 2, Section 1, contains the results about ACCPL condition in skew PBW extensions over domains and Archimedian domains. In Section 2 we present several results about condition ACCP in skew PBW extensions over Σ-rigid rings (these rings were defined by Reyes [38]). Finally, we present some conclusions and a possible future line of research.

[1]The results presented in this chapter have been submitted to publication.

CHAPTER 1

Duo and quasi-duo over skew PBW extensions

In this chapter we present the necessary notions for the development of this work and some conclusions about the properties duo and quasi-duo over the skew PBW extensions.

1.1 Basic notions

We start defining the Ore extensions (of injective type) with the aim of showing why these extensions are particular examples of skew PBW extensions.

Definition 1.1.1 ([33]). Let R be a ring, σ a ring endomorphism of R, and δ an σ-derivation on R, i.e., δ is any additive map $\delta : R \to R$ such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. We shall write $S = R[x; \sigma, \delta]$ provided

(i) S is a ring, containing R as a subring;

(ii) x is an element of S;

(iii) S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(iv) $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

Such a ring S is called a *skew polynomial ring* over R, or an *Ore extension* of R. It says that $R[x; \sigma, \delta]$ is of *endomorphism type* if $\delta = 0$, and of *derivation type* if $\sigma = i_R$. An Ore extension is of *injective type* if σ is injective.

We will show the construction for an special type of iterated skew polynomial ring of endomorphism type. Suppose that $\sigma_1, \ldots, \sigma_n$ are commuting endomorphisms of a ring R (that is, $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all i, j). First, set $S_1 = R[x_1; \sigma_1]$, σ_2 extends uniquely to an endomorphism $\widehat{\sigma_2}$ of S_1 such that $\widehat{\sigma_2}(x_1) = x_1$, and set $S_2 = S_1[x_2; \widehat{\sigma_2}]$. Similarly, once S_i has been constructed for some $i < n$, we build $S_{i+1} = S_i[x_{i+1}; \widehat{\sigma_{i+1}}]$, where $\widehat{\sigma_{i+1}}$ is the unique endomorphism of S_i such that $\widehat{\sigma_{i+1}} \mid_R = \sigma_{i+1}$ and $\widehat{\sigma_{i+1}}(x_j) = x_j$ for $j = 1, \ldots, i$. Finally, let $S = S_n = R[x_1; \sigma_1][x_2; \widehat{\sigma_2}] \cdots [x_n; \widehat{\sigma_n}]$. A standard notation for S, is

$S = R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$. Note that $x_i x_j = x_j x_i$ for all i, j, and that $x_i r = \sigma_i(r) x_i$ for all i and all $r \in R$.

Analogously it is possible to build an iterated skew polynomial ring of type derivation $S = R[x_1, \ldots, x_n; \delta_1, \ldots, \delta_n]$, and an iterated skew polynomial ring $S = R[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$.

Example 1.1.2. Let $q \in k$ (with k an arbitrary field) be any nonzero scalar such that $q \neq \pm 1$. The *quantized enveloping algebra* of $\mathfrak{sl}_2(k)$ (special linear Lie algebra, which consist of 2×2 matrices over k having trace 0) corresponding to the choice of q is the k-algebra $U_q(\mathfrak{sl}_2(k))$ presented by four generators E, F, K, K^{-1} and five relations

$$KK^{-1} = K^{-1}K = 1 \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

$$KE = q^2 EK \qquad KF = q^{-2} FK.$$

$U_q(\mathfrak{sl}_2(k))$ can be expressed as an iterated skew polynomial ring of the form

$$k[K^{\pm 1}][E; \sigma_1][F; \sigma_2, \delta_2].$$

Example 1.1.3. Given any $q \in k$, the corresponding *quantized coordinate ring* of $M_2(k)$ is the k-algebra $\mathcal{O}_q(M_2(k))$ presented by four generators x_{11}, x_{12}, x_{21}, x_{22} and the six relations

$$x_{11}x_{12} = qx_{12}x_{11} \qquad x_{12}x_{22} = qx_{22}x_{12}$$

$$x_{11}x_{21} = qx_{21}x_{11} \qquad x_{21}x_{22} = qx_{22}x_{21}$$

$$x_{12}x_{21} = x_{21}x_{12} \qquad x_{11}x_{22} - x_{22}x_{11} = (q - q^{-1})x_{12}x_{21}.$$

This algebra is also called *the coordinate ring of quantum* 2×2 *matrices over* k, or *the* 2×2 *quantum matrix algebra over* k. $\mathcal{O}_q(M_2(k))$ can be expressed as an iterated skew polynomial ring of the form $k[x_{11}][x_{12}; \sigma_{12}][x_{21}; \sigma_{21}][x_{22}; \sigma_{22}, \delta_{22}]$.

Now we will remember the definition given by Gallego and Lezama [9] of the structure of greater interest in this work, the skew Poincaré-Birkhoff-Witt extensions or skew PBW extensions. As we will see in Proposition 1.1.5 they are a type of non-commutative rings more general than Ore extensions of injective type.

Definition 1.1.4 ([9], Definition 1)**.** Let R and A be rings. We say that A is *a skew PBW extension of* R (also called a σ-PBW extension of R), which is denoted by $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$, if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist elements $x_1, \ldots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ $(x^0 := 1)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements x_i, x_j with $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \qquad (1.1.1)$$

The next proposition establishes the analogy between Ore extensions of injective type, i.e., σ_i is injective for all $1 \leq i \leq n$, and skew PBW extensions.

Proposition 1.1.5 ([9], Proposition 3)**.** *Let A be a skew PBW extension of R. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and an σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$.*

Suárez and Reyes [48] presented some examples of skew PBW extensions according to the next definition. For now, we remark the relation between the notions of constant skew PBW extension and Σ-rigid rings (see Example 2.2.2).

Definition 1.1.6 ([48], Definition 2.3)**.** Let A be a skew PBW extension of R, $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta := \{\delta_1, \ldots, \delta_n\}$, where σ_i and δ_i $(1 \leq i \leq n)$ are as in Proposition 1.1.5.

(a) A is called *pre-commutative* if the conditions (iv) in Definition 1.1.4 are replaced by: For any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in Rx_1 + \cdots + Rx_n. \tag{1.1.2}$$

(b) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 1.1.4 are replaced by

(iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i; \tag{1.1.3}$$

(iv') for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \tag{1.1.4}$$

(c) A is called *bijective*, if σ_i is bijective for each $\sigma_i \in \Sigma$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(d) If $\sigma_i = \mathrm{id}_R$ for every $\sigma_i \in \Sigma$, we say that A is a skew PBW extension of *derivation type*.

(e) If $\delta_i = 0$ for every $\delta_i \in \Delta$, we say that A is a skew PBW extension of *endomorphism type*.

(f) Any element r of R such that $\sigma_i(r) = r$ and $\delta_i(r) = 0$ for all $1 \leq i \leq n$, will be called a *constant*. A is called *constant* if every element of R is constant.

(g) A is called *semi-commutative* if A is quasi-commutative and constant.

Recall that a ring B is called $\mathbb{Z}$-graded, if there exists a family of subgroups $B_n, n \in \mathbb{Z}$, of B such that $B = \bigoplus_n B_n$ (as abelian groups), and $B_n \cdot B_m \subseteq B_{n+m}$ for all n, m; a graded ring B is called $\mathbb{N}$-graded if $B_n = 0$ for all $n < 0$. Let $B = \bigoplus_n B_n$ and $A = \bigoplus_n A_n$ be graded rings. A ring homomorphism $\varphi : A \to B$ is called a *graded ring homomorphism* if $\varphi(A_n) \subset B_n$ for all $n \in \mathbb{Z}$.

A *filtered ring* is a ring B with a family $FB = \{F_nB \mid n \in \mathbb{Z}\}$ of additive subgroups of B, where we have the ascending chain $\cdots \subset F_{n-1}B \subset F_nB \subset \cdots$ such that $\bigcup_{n\in\mathbb{Z}} F_nB = B$, $1 \in F_0B$ and $F_nBF_mB \subseteq F_{n+m}B$, for all $n, m \in \mathbb{Z}$. From a filtered ring B, it is possible to construct its associated graded ring $G(B)$ taking $G(B)_n := F_nB/F_{n-1}B$.

Definition 1.1.7 ([9], Definition 6)**.** If A is a skew PBW extension of R, then:

(i) for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$, $\deg(X) := |\alpha|$, and $X_0 := 1$. The symbol $\succeq$ will denote a total order defined on $\mathrm{Mon}(A)$ (a total order on $\mathbb{N}^n$). For an element $x^\alpha \in \mathrm{Mon}(A)$, $\exp(x^\alpha) := \alpha \in \mathbb{N}^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1X_1 + \cdots + a_mX_m$, with $a_i \in R$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_mX_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see Lezama and Gallego ([9], Section 3).

The next result establishes that a skew PBW extensions is a filtered ring and computes explicitly its associated graded ring this ring will be involved in some of the main theorems of Chapter 2, allowing a characterization for the PBW extensions that are ascending chain condition on principal left ideals domain (ACCPL-domain) (see Definition 1.3.10) and right (left) Archimedian domain (see Definition 2.1.9).

Proposition 1.1.8 ([24], Theorem 2.2)**.** *If A is a skew PBW extension of R, then A is a filtered ring with increasing filtration given by*

$$F_mA := \begin{cases} R, & \text{if } \ m = 0 \\ \{f \in A \mid \deg(f) \leq m\}, & \text{if } \ m \geq 1 \end{cases}$$

and the corresponding graded ring $G(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $G(A)$ is a quasi-commutative bijective skew PBW extension of R.

Since the notion of $\mathbb{Z}$-graded ring is of remarkable important for the work developed by Leroy, Matczuk and Puczyłowski [22], next we consider the work of Suárez [45] where he defined a graded skew PBW extension (see [45], Examples 2.8 and 2.9 for several examples).

Definition 1.1.9 ([45], Definition 2.6)**.** Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension of an $\mathbb{N}$-graded $\mathbb{K}$-algebra R. We said that A is *a graded skew PBW extension* if the following conditions hold:

(i) $x_1, \ldots, x_n$ have degree 1 in A.

(ii) σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation for all $1 \leq i \leq n$, where σ_i and δ_i are as in Proposition 1.1.5.

(iii) $x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in (1.1.1) and $c_{i,j} \in R_0$.

The following notation is necessary for the accounts in the proofs of the Theorems 2.1.4, 2.1.10 and 2.2.5.

Proposition 1.1.10 ([9], Theorem 7)**.** *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(i) *for each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

(ii) *For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.*

In the noncommutative setting an *integral domain*, briefly called a *domain*, is defined as a ring in which the product of any two nonzero elements is nonzero. Proposition 1.1.11 establishes that skew PBW extensions over domains are domains.

Proposition 1.1.11 ([24], Proposition 4.1)**.** *Let A be a skew PBW extension of a ring R. If R is a domain, then A is also a domain.*

For an Ore extension $R[x; \sigma, \delta]$ of R, if there exists $d \in R$ such that $\delta(r) = dr - \sigma(r)d$, for all $r \in R$, then δ is called an *inner σ-derivation* of R. If this is the case, one can show that $R[x; \sigma, \delta] = R[x - d; \sigma]$. The generalization of this fact for skew PBW extensions is formulated in the following proposition which provides a characterization for skew PBW extensions that are Archimedean domains (see Corollary 2.1.14).

Proposition 1.1.12 ([23], Proposition 2.5)**.** *Let A be a skew PBW extension of a ring R. If, for every $1 \leq i \leq n$, δ_i is inner, then A is a skew PBW extension of R of endomorphism type.*

Proposition 1.1.13 and Remark 1.1.14 are of great relevance for the proof of the Theorem 2.2.5.

Proposition 1.1.13 ([38], Proposition 2.9)**.** *If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and r is an element* *of R, then*

$$
\begin{aligned}
x^{\alpha}r = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}r = x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}\left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n-j}\delta_n(\sigma_n^{j-1}(r))x_n^{j-1}\right)\\
+ x_1^{\alpha_1}\cdots x_{n-2}^{\alpha_{n-2}}\left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j}\delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r)))x_{n-1}^{j-1}\right)x_n^{\alpha_n}\\
+ x_1^{\alpha_1}\cdots x_{n-3}^{\alpha_{n-3}}\left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j}\delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r))))x_{n-2}^{j-1}\right)x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}\\
+ \cdots + x_1^{\alpha_1}\left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2-j}\delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r))))))x_2^{j-1}\right)x_3^{\alpha_3}x_4^{\alpha_4}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}\\
+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r))))x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \qquad \sigma_j^0 := \mathrm{id}_R \quad \text{for} \quad 1 \le j \le n.
\end{aligned}
$$

Remark 1.1.14. About Proposition 1.1.13, we have the following observation: if $X_i := x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$ and $Y_j := x_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}}$, when we compute every summand of $a_iX_ib_jY_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i. This assertion follows from the expression:

$$
\begin{aligned}
a_iX_ib_jY_j = a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + a_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
+ a_ix_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
+ a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
+ \cdots + a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b_j)}x_n^{\alpha_{in}}x^{\beta_j}\\
+ a_ix_1^{\alpha_{i1}}\cdots x_{i(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b_j}x^{\beta_j}.
\end{aligned}
$$

1.2 Duo rings

A ring R is called *right (left) duo* if every right (left) ideal of R is a two-sided ideal. A right and left duo ring is called a *duo ring*. A characterization of this type of rings was shown by Gary F. Birkenmeier y Ralph P. Tucci ([3], Proposition 6), where R is a right duo ring if and only if R/T is strongly right bounded for all ideals T of R. Courter ([4], Corollary 2.3) proved that a right duo ring with unity element which is a finite dimensional algebra over an arbitrary field is a duo ring. However Y. Hirano, C.-H. Hong, J.-Y. Kim and J.K. Park extend it ([13], Theorem 3) where R is a right artinian ring which is module finite over its center. He also proved ([13], Lemma 3) that an ordinary polynomial ring is right duo only if it is commutative. This result was extended by Marks [26] for the skew polynomial ring $S = R[x;\sigma]$, that is, if S is right (left) duo, R must be commutative and σ must be the identity.

The paper of Marks (see [26]) is one of the most important references in this work. Next we present some of its main results.

Proposition 1.2.1 ([26], Theorem 1)**.** *If the skew polynomial ring $R[x;\sigma]$ is left or right duo, then $R[x;\sigma]$ is commutative.*

Proof. We follow the proof of [26]. Put $S = R[x;\sigma]$. Since S is one-sided duo, it must be Dedekind-finite, i.e., if $fg = 1$ then $gf = 1$, for $f, g \in S$. We will begin by assuming, for a contradiction, that σ is not the identity automorphism of R. In this case, there exists some $a \in R$ such that $\sigma(a) \neq a$. Let $S(1 + ax + x^2)$ left ideal of S, see that $(1+ax+x^2)x = x+ax^2+x^3 \notin S(1+ax+x^2)$. Suppose that $(1+ax+x^2)x \in S(1+ax+x^2)$, i.e.,$(1 + ax + x^2)x = f(1 + ax + x^2)$ for $f \in S$ so,

$$\begin{aligned}(1 + ax + x^2)x =& f(1 + ax + x^2) = (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)(1 + ax + x^2)\\ =& b_0 + b_0ax + b_0x^2 + b_1x + b_1xax + b_1x^3 + \cdots + b_nx^n + b_nx^nax + b_nx^{n+2}\\ =& b_0 + b_0ax + b_0x^2 + b_1x + b_1\sigma(a)x^2 + b_1x^3 + \cdots\\ &+ b_nx^n + b_n\sigma^n(a)x^{n+1} + b_nx^{n+2}\end{aligned}$$

Comparing similar terms, observe that, $b_0 = b_2 = \cdots = b_n = 0$, $b_0a + b_1 = 1$, i.e., $b_1 = 1$ and $b_0 + b_1\sigma(a) + b_2 = a$, i.e., $\sigma(a) = a$ contradiction. So S is not left duo.

Thus, S must be right duo. If σ were injective, then for $f, g \in S \setminus \{0\}$ with f monic, $\deg(fg) = \deg f + \deg g$; let $(1 + ax + x^2)S$ right ideal of S, suppose that $x(1 + ax + x^2) \in (1 + ax + x^2)S$, i.e., $x(1 + ax + x^2) = x + \sigma(a)x^2 + x^3 = (1 + ax + x^2)f$ with $\deg(f) = 1$ so

$$\begin{aligned}x(1 + ax + x^2) =& x + \sigma(a)x^2 + x^3 = (1 + ax + x^2)(b_0 + b_1x)\\ =& b_0 + axb_0 + x^2b_0 + b_1x + axb_1x + x^2b_1x\\ =& b_0 + a\sigma(b_0) + \sigma^2(b_0)x^2 + b_1 + a\sigma(b_1)x^2 + \sigma^2(b_1)x^3\end{aligned}$$

Comparing similar terms, observe that, $b_0 = 0$, $a\sigma(b_0) + b_1 = 1$, i.e., $b_1 = 1$ and $\sigma^2(b_0) + a\sigma(b_1) = \sigma(a)$, i.e., $\sigma(a) = a$ contradiction. So

$$x(1 + ax + x^2) = x + \sigma(a)x^2 + x^3 \notin (1 + ax + x^2)S.$$

Thus, since S is right duo, σ must be surjective ($Sx \subsetneq xS$) but not injective. Now we show that only units are carried to units by σ. Let $U(R)$ the set of the units of R, suppose $a \in R$ is such that $\sigma(a) \in U(R)$. Then because S is right duo,

$$\sigma(a)x = xa \in aS \Rightarrow \sigma(a) \in aR \Rightarrow a \in U(R).$$

Therefore $\sigma^{-1}(U(R)) = U(R)$. Since σ is surjective but not injective, there exist nonzero elements c_0, $c_1 \in R$ such that $\sigma(c_1) = 0$ and $\sigma(c_0) = c_1$. Now, since S is right duo,

$$x(c_0 + c_1x + x^2) = c_1x + x^3 \in (c_0 + c_1x + x^2)S.$$

Write

$$c_1x + x^3 = (c_0 + c_1x + x^2)(d_0 + d_1x + \cdots + d_nx^n). \tag{1.2.1}$$

Comparing x^0-, x^1-, and x^3-coefficients in Eq. (1.2.1) we obtain the following three equations:

$$c_0d_0 = 0 \tag{1.2.2}$$

$$c_0d_1 + c_1\sigma(d_0) = c_1, \tag{1.2.3}$$

$$c_0d_3 + c_1\sigma(d_2) + \sigma^2(d_1) = 1. \tag{1.2.4}$$

By Eq. (1.2.2), we have $0 = \sigma(c_0d_0) = c_1\sigma(d_0)$. Thus, by Eq. (1.2.3),

$$c_0d_1 = c_1. \tag{1.2.5}$$

Applying σ^2 to Eq. (1.2.4), we obtain $\sigma^4(d_1) = 1$; so, since $\sigma^{-1}(U(R)) = U(R)$, we conclude that $\sigma(d_1)$ is a unit. But now, applying σ to Eq. (1.2.5), we obtain $c_1\sigma(d_1) = 0$, which contradicts our choice of $c_1 \neq 0$. Consequently, σ is the identity automorphism of R, and we can write $S = R[x]$, which by hypothesis is one-sided duo. Let $a, b \in R$ be arbitrary. If S is left duo, then $(a + x)b \in S(a + x) \to (a + x)b = b(a + x) \to ab = ba$; if S is right duo, then $b(a + x) \in (a + x)S \to b(a + x) = (a + x)b \to ba = ab$. In either case, R must be commutative. □

As in the skew PBW extensions the endomorphisms turn out to be injective, the previous proposition allows us to say that the skew PBW extensions of type endomorphism do not has the right duo property and therefore they are not duo rings. In the same way, the following proposition allows us to discard the left duo property and therefore limits our analysis to the property right duo. Nevertheless, we will consider important to present the more remarkable results about the study of this property for Ore extensions.

Proposition 1.2.2 ([26], Theorem 2). *If $S = R[x; \sigma, \delta]$ is left duo, then S is commutative.*

Proof. If $\delta(r) \neq 0$ for some $r \in R$, then $xr \notin Sx$, contrary to hypothesis. Hence δ is the zero map, and we can apply Proposition 1.2.1. □

We observe in the next propositions that the right duo hypothesis on $R[x; \sigma, \delta]$ imposes some restrictions on the endomorphism and the derivation. Following Marks [26], we recall that an ideal I of R is called a *σ-ideal*, if $\sigma(I) \subseteq I$, and it is called a *δ-ideal*, if $\delta(I) \subseteq I$; I is called a *(σ, δ)-ideal*, if both containments hold. The (σ, δ)-ideals impose one of the necessary conditions for the Ore extension $S = R[x; \sigma; \delta]$ be right duo.

Proposition 1.2.3. *If $S = R[x; \sigma, \delta]$ is right duo. Then:*

(1) ([26], Lemma 3). *For any $i \in \mathbb{N}$ we have $ker(\sigma^i) \subset J(R)$, where $J(R)$ is the Jacobson radical of the ring R..*

(2) ([26], Theorem 4). *If S is not commutative, then $(0) \neq ker\ \sigma \subset J(R)$.*

(3) ([26], Lemma 7). *Every right ideal of R is a (σ, δ)-ideal.*

(4) ([26], Proposition 8). *If $e \in R$ is any idempotent, e is a central idempotent, $\delta(e) = 0$, and $\sigma(e) = e$.*

(5) ([26], Corollary 11). *$r \in J(R) \Leftrightarrow \sigma(r) \in J(R)$, and $r \in U(R) \Leftrightarrow \sigma(r) \in U(R)$.*

Proposition 1.2.4 ([26], Theorem 10). *Define the ideal $N = \bigcup_{i=1}^{\infty} ker(\sigma^i) \subset R$. If $S = R[x; \sigma, \delta]$ is right duo, then the following conditions hold:*

(1) *The ideal N is a (σ, δ)-ideal contained in $J(R)$, and $N \neq (0)$ except in the trivial case where S is commutative.*

(2) *For any $r \in R$, the sequence $\{\sigma^n(r)\}_{n\in\mathbb{N}}$ is eventually constant.*

(3) *For any $r \in R$, the sequence $\{\sigma^n(\delta(r))\}_{n\in\mathbb{N}}$ is eventually zero.*

(4) *The factor ring S/NS is isomorphic to the commutative polynomial ring $(R/N)[x]$.*

The next example meets the conditions given by Marks but is not right duo ring.

Example 1.2.5 ([26], Example 13)**.** Let T be any ring, and let M be any nonzero (T,T)-bimodule. Put $R = T \oplus M$, with addition defined componentwise, and multiplication defined by $(t_1, m_1)(t_2, m_2) = (t_1t_2, t_1m_2 + m_1t_2)$. Define $\sigma : R \to R$ and $\delta : R \to R$ by

$$\sigma(t, m) = (t, 0),\ \delta(t, m) = (0, m) \text{ for all } t \in T,\ m \in M.$$

Then σ is an endomorphism, δ is a σ-derivation, the conclusions of Proposition 1.2.3 and (1)through (3) of Proposition 1.2.4 all hold, and

$$ker\ \sigma = \bigcup_{i=1}^{\infty} ker(\sigma^i) = 0 \oplus M \subseteq J(R).$$

(If T is commutative, then (iv) of Proposition 1.2.4 and the conclusions of Proposition 1.2.3 also hold.) Choose any nonzero $m \in M$. If there were to exist elements $(t_i, m_i) \in R$ such that

$$x\big((1, m) + (-1, m)x\big) = \big((1, m) + (-1, m)x\big)\left(\sum_{i=0}^{n}(t_i, m_i)x^i\right) \in R[x; \sigma, \delta],$$

then comparing x^0-coefficients would yield $(0, m) = (t_0, mt_0)$, contradicting $m \neq 0$. Thus, $R[x; \sigma, \delta]$ is not right duo.

Matczuk [28] shows that noncommutative Ore extensions $R[x; \sigma, \delta]$ which are right duo rings do exist and that the necessary conditions obtained by Marks are not sufficient for the Ore extension $S = R[x; \sigma, \delta]$ to be right duo. He uses the unital split corner ring and then define the skew derivations on these to show that there are Ore extensions that are right duo (see Proposition 1.2.17). Let $R^\sigma = \{r \in R \mid \sigma(r) = r\}$; observe that statements (2) and (3) of Proposition 1.2.4 say that R^σ is *a unital split corner ring* of a ring R, i.e., R^σ is a unital subring of R, $R = R^\sigma \oplus N$ as abelian groups and N is an ideal of R. The maps σ and δ satisfy: for any $r \in N$, there exists $n \in \mathbb{N}$ such that $\sigma^n(r) = 0$ and $\delta(R) \subseteq N$.

Proposition 1.2.6 ([28], Proposition 1.5)**.** *Let R be either a left or a right Noetherian ring. Suppose that $R[x; \sigma, \delta]$ is a right duo ring which is noncommutative. Then there exists a noncommutative Ore extension $R'[x; \sigma', \delta']$, which is a right duo ring, such that:*

(i) *$R' = A \oplus M$ where A is a unital split corner subring of R' with $M^2 = 0$ and $M \neq 0$.*

(ii) *$\sigma' : R' \to R'$ is defined by $\sigma'(a + l) = a$, for any $a \in A$ and $l \in M$, and $\delta'(R') \subseteq M$.*

Moreover, R' can be taken to be a factor ring of R.

In this proposition we show that there are indeed corner extensions which are right duo rings, and therefore in Subsection 1.2.1 its relationship with the Ore extension $R[x;\sigma,\delta]$, here A is a unital split corner subring of $R = A \oplus M$, with $M^2 = 0$.

Proposition 1.2.7 ([28], Theorem 2.4)**.** *Let A be a right duo ring and $R = A \oplus M$, where M is an (A, A)-bimodule such that M is faithful (i.e. for every module U, $M \times U = 0$ implies $U = 0$) as a left A-module and simple as a right A-module. Then:*

(1) *R is a right duo ring.*

(2) *R is left duo iff M is faithful as a right A-module (i.e., A is a division ring) and simple as a left A-module.*

For a subset S of an (R, R)-bimodule M, $l.ann_R(S)$ will stand for the left annihilator of S in R, i.e., $l.ann_R(S) = \{r \in R \mid rS = 0\}$. The right annihilator $r.ann_R(S)$ is defined similarly. The coming proposition helps us calculate the skew derivations of $A \oplus M$ (see Proposition 1.2.10)

Proposition 1.2.8 ([28], Lemma 2.6)**.** *Let A be a right duo ring. The following conditions are equivalent:*

(1) *There exists an (A, A)-bimodule M such that M is faithful as left A-module and simple as right A-module.*

(2) *There exist a right primitive ideal P of A and an injective homomorphism $\phi : A \to A/P$.*

Sketch of the proof.

(1)⇒(2) Let P denote the annihilator of M_A. Then P is right primitive ideal of A and A/P is a division ring as A is a right duo ring. This means that, for any $0 \neq m \in M, r.ann_A(m) = P$. Let us fix $0 \neq m \in M$ and consider M as $(A, A/P)$-bimodule. Then $M = m(A/P)$ and for any $a \in A$, $am = m\phi_m(a)$ for a suitable element $\phi_m(a) \in A/P$. Notice that, because $r.ann_{A/P}(m) = 0$, the element $\phi_m(a)$ is uniquely determined by a. Thus we have a well defined map $\phi = \phi_m : A \to A/P$. ϕ is a ring homomorphism. If $\phi(a) = 0$, then $0 = m\phi(a)(A/P) = am(A/P) = aM$. Hence $a = 0$ follows, as the left A-module ${}_AM$ is faithful. This shows that ϕ is injective.

(2)⇒(1) Let P be a right primitive ideal of R and $\phi : A \to A/P$ an injective homomorphism. Then, as A is right duo, A/P is a division ring. Let M be the one-dimensional right vector space over A/P. Let us fix $0 \neq m \in M$ and define left A module structure on M by setting $a \cdot (mr) = m\phi(a)r$, for any $a \in A$ and $r \in A/P$. This determines an $(A, A/P)$-bimodule structure on M. Notice that if $aM = 0$, then $m\phi(a) = 0$ and $a = 0$ follows, as ϕ is injective and $r.ann_{A/P}(m) = 0$. This induces the desired (A, A)-bimodule structure on M.

□

Remark 1.2.9. A will stand for a commutative domain, P for a maximal ideal of A, K will denote the field A/P and $\phi : A \to K$ a fixed injective homomorphism of rings. For any element $a \in A$, $\overline{a}$ will denote the canonical image of a in $K = A/P$. By Proposition 1.2.8, the right K vector space vK with the basis $\{v\}$ has a structure of (A, A)- bimodule given by $a \cdot vk = v\phi(a)k$ and $vk \cdot a = vka$, for any $a \in A$ and $k \in K$. Then vK is faithful as a left A-module and simple as a right A-module. Thus, by Proposition 1.2.7, $R = A \oplus vK$ is a right duo ring. From now on, $\sigma : R \to R$ stands for the endomorphism of R given by $\sigma(a + v\bar{l}) = a$, for any $a, l \in A$.

Proposition 1.2.10. (1) ([28], Lemma 3.1). *Let d_y denote the inner σ-derivation of R determined by the element $y = c + v\overline{m} \in A \oplus vK = R$, where $c, m \in A$. Then:*

- (i) *$d_y(a + v\bar{l}) = v\phi(c)\bar{l} + v\overline{m}(\overline{a} - \phi(a)) \in vK$, for any $a + v\bar{l} \in R$. In particular, $d_y(v) = v\phi(c)$.*
- (ii) *If $d_y(A) = 0$, then $d_y(a + v\bar{l}) = v\phi(c)\bar{l}$, for any $a + v\bar{l} \in R$.*

(2) ([28], Lemma 3.2). *For any $w \in K$,define $\delta_w : R \to R$ by setting $\delta_w(a + v\bar{l}) = vw\bar{l}$, for any $a, l \in A$, δ_w is a σ-derivation of $R = A \oplus vK$. Moreover δ_w is an inner σ-derivation iff $\overline{w} \in \phi(A)$.*

The previous proposition shows a characterization of the inner σ-derivations of $A \oplus vK$ as a consequence of the Remark 1.2.9, also how each element of K determines an inner derivation of $A \oplus vK$. Therefore, the following proposition gives a description of all σ-derivations of R.

Proposition 1.2.11 ([28], Theorem 3.3)**.** *Let δ be a nonzero σ-derivation of $R = A \oplus vK$. Then:*

(1) *There exists $w \in K$ such that $\delta(v) = vw$.*

(2) *If $\delta(vK) = 0$, then one of the following conditions holds:*

- (i) *δ is an inner σ- derivation of R;*
- (ii) *$\phi = id_K$, i.e., $R = K \oplus vK$ is a commutative ring, δ is an outer σ-derivation and there exists a derivation d of the field K such that $\delta(a + vb) = vd(a)$, for any $a, b \in K$.*

(3) *Let $w \in K$ be such that $\delta(v) = vw$. Then $(\delta - \delta_w)(vK) = 0$, i.e., $\delta - \delta_w$ is a σ-derivation satisfying the assumption of the statement* 2.

From above proposition is obtained the following classification of Ore extension $R[x; \sigma, \delta]$ over our ring $R = A \oplus vK$:

Proposition 1.2.12 ([28], Proposition 3.7)**.** *Let δ be a σ-derivation of $R = A \oplus vK$. Then:*

(1) *Suppose that R is noncommutative. Then $R[x; \sigma, \delta]$ is R-isomorphic either to $R[x; \sigma]$ or to $R[x; \sigma, \delta_w]$, for some $w \in K$ $\phi(A)$, where $\delta_w(a + v\bar{l}) = vw\bar{l}$, for any $a, l \in A$.*

(2) *Suppose that R is commutative. Then $R = K \oplus vK$ and $R[x; \sigma, \delta]$ is R-isomorphic either to $R[x; \sigma]$ or to $R[x; \sigma, \widehat{\delta}]$, where $\widehat{\delta}(a + vb) = vd(a)$, for any $a, b \in K$, and d denotes a nonzero derivation of K.*

1.2.1 Ore extensions which are right duo rings

Here we present the characterization of the Ore extensions $R[x;\sigma,\delta]$ that are right duo through the derivation defined above and taking $R = K \oplus vK$ presented in Proposition 1.2.17. The following results allow us to prove this characterization. First we define certain elements that allow us to characterize in the Proposition 1.2.14 the bilateral ideals in $R[x;\sigma,\delta_w]$.

Definition 1.2.13 ([28], Definition 4.3). For any polynomial $f = \sum_{k=0}^{n}(a_k + v\bar{l}_k)x^k \in R[x;\sigma,\delta_w]$ we set:

(1) $f_A = \sum_{k=0}^{n} a_k x^k \in A[x] \subseteq R[x;\sigma,\delta_w]$ and $f_v = f - f_A$.

(2) $D_f = \sum_{k=0}^{n} \phi(a_k)\overline{w}^k \in K$, that is $D_f = \phi_w(f_A)$.

Proposition 1.2.14 ([28], Proposition 4.5). *For a polynomial $f \in R[x;\sigma,\delta_w]$, the following conditions are equivalent:*

(1) *$fR[x;\sigma,\delta_w]$ is a two-sided ideal of $R[x;\sigma,\delta_w]$.*

(2) *One of the following conditions holds:*

- (i) *$D_f \neq 0$;*
- (ii) *$f_A = 0$, i.e., $f \in vR[x;\sigma,\delta_w]$;*
- (iii) *$vf = 0$ and $f = f_A$.*

A direct consequence of the previous proposition is:

Proposition 1.2.15 ([28], Corollary 4.6). *Let $w \in K$ and $\widehat{\phi(A)}$ denote the subfield of K generated by $\phi(A)$. Then:*

(1) *If w is transcendental over $\widehat{\phi(A)}$ then $fR[x;\sigma,\delta_w]$ is a two-sided ideal of $R[x;\sigma,\delta_w]$, for any $f \in R[x;\sigma,\delta_w]$, i.e., $R[x;\sigma,\delta_w]$ is a right duo ring.*

(2) *If w is algebraic of degree $n+1$ over $\widehat{\phi(A)}$, for some $n \geq 0$, then:*

- (i) *for every polynomial $f \in R[x;\sigma,\delta_w]$ of degree $deg(f) \leq n$, $fR[x;\sigma,\delta_w]$ is a two-sided ideal of $R[x;\sigma,\delta_w]$;*
- (ii) *there exists a polynomial $f \in R[x;\sigma,\delta_w]$ of degree $n+1$ such that $fR[x;\sigma,\delta_w]$ is not a two-sided ideal of $R[x;\sigma,\delta_w]$.*

When $P = 0$, $R = K \oplus vK$, then $vf \neq 0$, for any polynomial $f \in R[x;\sigma,\delta_w]$ with $f_A \neq 0$. Thus, of Proposition 1.2.14 we obtain:

Proposition 1.2.16 ([28], Corollary 4.7). *Suppose $R = K \oplus vK$ and $f \in R[x;\sigma,\delta_w]$. Then $fR[x;\sigma,\delta_w]$ is a two-sided ideal of $R[x;\sigma,\delta_w]$ iff either $D_f \neq 0$ or $f \in vR[x;\sigma,\delta_w]$.*

The proof of the following proposition is obtained from the Propositions 1.2.12 and 1.2.15, this proof can be seen in [27].

Proposition 1.2.17 ([28], Theorem 4.8). *Let A be a commutative domain with a maximal ideal P, $\phi : A \to A/P = K$ an injective homomorphism and $R = A \oplus vK$ the associated unital split corner extension of A. Then the following statements are equivalent:*

(1) $R[x; \sigma, \delta]$ *is a right duo ring;*

(2) *There exists $w \in K$ such that w is transcendental over the subfield of K generated by $\phi(A)$ and $R[x; \sigma, \delta]$ is R-isomorphic to $R[x; \sigma, \delta_w]$.*

Finally, it is possible to affirm that the non-commutative Ore extension $B[x; \tau, \delta]$ with coefficients in a Noetherian ring is never duo ring:

Proposition 1.2.18 ([28], Proposition 4.11). *Let B be a commutative Noetherian ring. If the Ore extension $B[x; \tau, \delta]$ is a right (left) duo ring, then $B[x; \tau, \delta] = B[x]$ is a commutative polynomial ring.*

Sketch of the proof. If $B[x; \tau, \delta]$ is left duo, then the thesis is a consequence of Proposition 1.2.2. Suppose that $B[x; \tau, \delta]$ is a right duo ring which is noncommutative. Then, by Proposition 1.2.6, there exists a noncommutative Ore extension $R[x; \sigma, \delta]$ which is right duo. Since R is a factor ring of B, R is commutative and Noetherian. Then, there is an ideal J of R such that $R/J \simeq A' \oplus M'$, where M' is an (A', A')-bimodule which is simple as a right A'-module and faithful as a left A'-module. Proposition 1.2.3 guarantees that J is a (σ, δ)-ideal, so $R[x; \sigma, \delta]/(JR[x; \sigma, \delta]) \simeq (R/J)[x; \sigma, \delta]$, we may assume that the commutative ring $R = A \oplus M$, where M is simple as a right A-module and faithful as a left A-module. Now, since R is commutative, Proposition 1.2.17 yields that $R[x; \sigma, \delta]$ is not right duo. This contradicts our assumption and completes the proof of the proposition. □

For the previous result, the Noetherianity of the ring would have to be eliminated to guarantee the duo property, and since in the skew PBW extensions the Noetherianity is one of the most important and studied properties, it is convenient (and sad) to leave aside the right duo property.

1.3 Quasi-duo rings

As we saw in the previous section it will not be very useful to study the property duo over skew PBW extensions. Consequently in this section we study a more general notion, *quasi-duo ring*, i.e., when every maximal one-sided ideal of a ring R is two-sided. So, each commutative ring is quasi-duo. We also review some results in the literature with the aim to study this property for skew PBW extensions.

Matczuk [28] established a necessary condition for an Ore extension to be quasi-duo ring. Later, Leroy, Matczuk and Puczyłowski [22] studied this property for $\mathbb{Z}$-graded rings, and they proved the characterization given in [21] using the graduation properties of the

Ore extension.

Matzuck presents what could be the first characterization of an Ore extension quasi-duo:

Proposition 1.3.1 ([28], Proposition 1.4)**.** *Let $S = R[x;\sigma,\delta]$. Suppose that R is a right duo ring, the factor ring S/NS is isomorphic to the commutative polynomial ring $(R/N)[x]$, where $N = \bigcup_{i=1}^{\infty} ker(\sigma^i) \subset R$ and every right ideal of R is a (σ,δ)-ideal. If N is a nil ideal of R, then S is a quasi-duo ring.*

Proof. We follow the proof of [28]. Let I be a nilpotent two-sided ideal of R. By assumption, I is (σ,δ)-stable, so $IR[x;\sigma,\delta]$ is also a nilpotent ideal of $R[x;\sigma,\delta]$. In particular, $IR[x;\sigma,\delta]$ is contained in the Jacobson radical $J(R[x;\sigma,\delta])$ of $R[x;\sigma,\delta]$. Let $a \in N$. Since R is a right duo ring and a is a nilpotent element, aR is a nilpotent two-sided ideal of R. Hence, by the above $NR[x;\sigma,\delta] \subseteq J(R[x;\sigma,\delta])$ follows. This implies that $NR[x;\sigma,\delta]$ is contained in any maximal one-sided ideal of $R[x;\sigma,\delta]$. Now, the thesis is an easy consequence of the fact that $R[x;\sigma,\delta]/(NR[x;\sigma,\delta]) \simeq (R/N)[x]$ is a commutative ring. □

We will denote by $\mathcal{A}$ the set of all maximal right ideals M of graded ring R such that $R_n \nsubseteq M$, for some $0 \neq n \in \mathbb{Z}$; $A(R) = \bigcap_{M\in\mathcal{A}} M$;

$$A_l = \{r \in R \mid R_n r \subseteq J(R), \text{ for every } 0 \neq n \in \mathbb{Z}\} \text{ and}$$

$$A_r = \{r \in R \mid rR_n \subseteq J(R), \text{ for every } 0 \neq n \in \mathbb{Z}\}.$$

The next proposition describe $A(R)$ in terms of A_l and A_r and have great relevance in the proof of the Proposition 1.3.4.

Proposition 1.3.2 ([22], Proposition 3 (i))**.** *Let R be a $\mathbb{Z}$-graded ring. Then $A(R) = A_l = A_r$.*

Proposition 1.3.3 ([22], Theorem 4)**.** *If a $\mathbb{Z}$-graded ring R is right (left) quasi-duo, then R/M is a field, for every $M \in \mathcal{A}$.*

Proposition 1.3.4 ([22], Theorem 5)**.** *A $\mathbb{Z}$-graded ring R is right (left) quasi-duo if and only if R_0 is right (left) quasi-duo and $R/A(R)$ is a commutative ring.*

Proof. We follow the proof of [22]. Suppose that R is right quasi-duo. Let M be a maximal right ideal of R_0. Clearly, MR is a proper right ideal of R. Consequently, MR is contained in a maximal right ideal T of R. Since R is right quasi-duo, $T \lhd R$. It is clear that $M = T \cap R_0$, so $M \lhd R_0$. Thus R_0 is a right quasi-duo ring. When $\mathcal{A} \neq \emptyset$, Proposition 1.3.3 implies that $R/A(R)$ is a subdirect sum of fields, so it is a commutative ring. If $\mathcal{A} = \emptyset$, then $A(R) = R$ and the ring $R/A(R)$ is also commutative. Suppose now that R_0 is right quasi-duo and $R/A(R)$ is commutative. Let I be the ideal of R generated by $\bigcup_{0\neq n\in\mathbb{Z}} R_n$. Then, by Proposition 1.3.2, $IA(R) \subseteq J(R)$. Hence $(I \cap A(R))^2 \subseteq J(R)$ and semiprimeness of $J(R)$ implies that $I \cap A(R) \subseteq J(R)$. This shows that $R/J(R)$ is a homomorphic image of a subdirect sum of rings R/I and $R/A(R)$. Clearly, R/I is a homomorphic image of R_0. Consequently, both R/I and $R/A(R)$ are right quasi-duo, so, further, $R/J(R)$ and R are right quasi-duo. When R is left quasi-duo, symmetric arguments apply. □

Thanks to the Definition 1.1.9, the previous proposition shows us how to find out if a graded skew PBW extension is or not quasi-duo. Of course it is relatively less tedious to see which ones are not, because due to the quotient presented there and the difficulty to describe the quotients in the PBW extensions, it is considered as future work to analyze this condition in these extensions and therefore give a complete list of the graded skew PBW extensions that are and are not quasi-duo.

We will give a proposition that is of great relevance in the proof of Proposition 1.3.6:

Proposition 1.3.5 ([22], Theorem 1)**.** *For every $\mathbb{Z}$-graded ring R:*

(i) *$J(R)$ is a homogeneous ideal, i.e., $J(R) = \bigoplus_{n\in\mathbb{Z}}(J(R)\cap R_n)$*

(ii) *If $r \in \bigcup_{0\neq n\in\mathbb{Z}} R_n$, then $1+r$ is invertible if and only if r is nilpotent.*

Let's consider $N(R)=\{r \in R \mid r\sigma(r)\cdots\sigma^n(r) = 0,$ for some positive integer $n\}$. Clearly, $N(R) = \{r \in R \subseteq R[x;\sigma] \mid (rx)^n = 0,$ for some positive integer $n\}$. Let $N(R)[x;\sigma]$ be the set of all polynomials from $R[x;\sigma]$ which have all their coefficients in $N(R)$. Notice also that $\sigma(N(R)) \subseteq N(R)$. Thus, if $N(R) \lhd R$, then $N(R)[x;\sigma] \lhd R[x;\sigma]$, σ induces an endomorphism, also denoted by σ, on $R/N(R)$ and $(R/N(R))[x;\sigma] \simeq R[x;\sigma]/N(R)[x;\sigma]$. The next proposition was presented by Leroy, Matczuk and Puczyłowski [22], they prove using $R[x;\sigma]$ as a $\mathbb{Z}$-graded ring.

Proposition 1.3.6 ([22], Lemma 8)**.** *Suppose that the skew polynomial ring $S = R[x;\sigma]$ is right (left) quasi-duo. Then $J(S) \subseteq N(R)[x;\sigma] \subseteq A(S)$.*

Sketch of the proof. Since S is right (left) quasi-duo, the ring $S/J(S)$ is reduced, $Rx^nN(R) \subseteq J(S)$, for all $n > 0$. The canonical $\mathbb{Z}$-graded of S together with the previous inclusion and the Proposition 1.3.2, $N(R)[x;\sigma] \subseteq A(S)$. Let $ax^n \in J(S)$, for some $n > 0$. Note that $ax^n \in S_n$ where S_n is the homogeneous component of degree n of S, thus $ax^n \in \bigcup_{0\neq n\in\mathbb{Z}} S_n$. Then, by Proposition 1.3.5, ax^n is nilpotent element of S since $1 + ax^n$ is invertible. Now see that x^na is also nilpotent element of S, let $m \in \mathbb{N}$ such that $(ax^n)^m = 0$:

$$\begin{aligned}(x^na)^{m+1} &= \underbrace{x^nax^na\cdots x^nax^na}_{m+1\text{-times}}\\ &= x^n(\underbrace{ax^na\cdots x^nax^n}_{m\text{-times}})a\\ &= x^n(ax^n)^m a = 0\end{aligned}$$

and so $x^na \in J(S)$. Hence $Rx^mx^{n-1}a \subseteq J(S)$, for all $m > 0$ and Proposition 1.3.2 shows that $x^{n-1}a \in J(S)$. Repeating this procedure, we obtain $xa \in J(S)$, this implies that $a \in N(R)$. Since $J(S)$ is a homogenous ideal, we obtain $J(R[x;\sigma]) \subseteq N(R)[x;\sigma]$. □

The necessary and sufficient conditions for the Ore extension of type endomorphism $R[x;\sigma]$ be quasi-duo are given by the following proposition.

Proposition 1.3.7 ([22], Corollary 9)**.** *$R[x;\sigma]$ is right (left) quasi-duo if and only if R is right (left) quasi-duo, $N(R) \lhd R$, $J(R[x;\sigma]) = J(R)\cap N(R) + N(R)[x;\sigma]x$, and $(R/N(R))[x;\sigma]$ is a commutative ring.*

During the development of this work Bien and Öinert [2] show how an Ore extension of derivation type with more than one indeterminate can never be quasi-duo. This result gives us more examples of skew PBW extensions that are not quasi-duo. Below the main results of this paper are presented.

Proposition 1.3.8 ([2], Theorem 1.1)**.** *Let $S = R[x;\delta]$ skew polynomial ring of type derivation and put $J_0 := J(S) \cap R$. The following five assertions are equivalent:*

(i) *S is left quasi-duo;*

(ii) *S is right quasi-duo;*

(iii) *Every left ideal of S containing the Jacobson radical $J(S)$ is two-sided, i.e. $S/J(S)$ is left duo;*

(iv) *Every right ideal of S containing the Jacobson radical $J(S)$ is two-sided, i.e. $S/J(S)$ is right duo;*

(v) *The quotient ring R/J_0 is commutative and $\delta(R) \subseteq J_0$.*

Before presenting the proof of Proposition 1.3.8 it is necessary the next result.

Proposition 1.3.9 ([2], Proposition 2.1)**.** *Let $S = R[x;\delta]$ be a skew polynomial ring of type derivation, and put $J_0 = J(S) \cap R$. If S is left (right) quasi-duo, then R/J_0 is commutative and $\delta(R) \subseteq J_0$.*

Next we present the proof of the Proposition 1.3.8.

Proof. We follow the ideas presented in [2]. We will show that (i) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (v) This implication follows from Proposition 1.3.9.

(v) $\Rightarrow$ (iii) Consider the morphism φ:

$$\varphi : R[x;\delta] \to (R/J_0)[x;\overline{\delta}], a_0 + a_1x + \cdots + a_nx^n \mapsto \overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n.$$

In our case, R/J_0 is commutative and $\overline{\delta} = 0$. Hence, $(R/J_0)[x, \delta]$ is commutative. Notice that φ is surjective and that $ker(\varphi) = J_0[x, \delta] \subseteq J(S)$. Hence, $S/(J_0[x, \delta]) \simeq (R/J_0)[x, \overline{\delta}]$ which is commutative. Therefore, every left ideal of S containing $J(S)$ is two-sided.

(iii) $\Rightarrow$ (i) This is trivial.

The proof of (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) is analogous. □

The objects described below play an important role in the Proposition 1.3.12, where it is stated that if the noncommutative skew polynomial ring $R[x;\delta]$ is PI (Definition 1.3.11) or meets the ascending chain condition on right annihilators, then it will not be quasi-duo. The Definition 1.3.10 is one of the most prominent in Chapter 2.

Definition 1.3.10. A ring R is said to satisfy *the ascending chain condition* (ACC) *on right annihilators* if there does not exist an infinite strictly ascending chain of right annihilators. A domain D is said to satisfy *the ascending chain condition on principal left ideals* (ACCPL) if there does not exist an infinite strictly ascending chain of principal left ideals of D. Domains satisfying *the ascending chain condition on principal right ideals* (ACCPR) are defined analogously. A domain D is called an *ACCP-domain* if it satisfies the ascending chain condition for principal ideals.

Definition 1.3.11. A *polynomial identity* (PI) on a ring R is defined as a polynomial $p(x_1, \dots, x_n)$ in non-commuting indeterminates $x_1, \dots, x_n$ with coefficients from $\mathbb{Z}$ such that $p(r_1, \dots, r_n) = 0$, for all $r_1, \dots, r_n \in R$. A *polynomial identity ring* (PI ring), is a ring R that satisfies some monic polynomial identity $p(x_1, \dots, x_n)$.

Proposition 1.3.12 ([2], Proposition 3.2)**.** *Let R be a ring satisfying* $\mathrm{Nil}(R) = 0$*. If R is a PI ring or satisfies the ascending chain condition on right annihilators, then the following two assertions are equivalent:*

(1) $R[x; \delta]$ *is quasi-duo;*

(2) $R[x; \delta]$ *is commutative.*

The most important result for this work obtained by Bien and Öinert, refers to differential polynomial ring in several indeterminates, following the notation of Bien and Öinert, let I be a non-empty (possibly infinite) countable set, let $D = \{\delta_i \mid i \in I\}$ be a family of derivations on R (by a "family" we mean that all δ_i's need not be distinct), and let $X = \{x_i \mid i \in I\}$ be a set of distinct non-commuting indeterminates. Given R, D and X, we can define the ring $R[X; D]$ which is the set of all polynomials in the indeterminates $x_i \in X$ with coefficients from R. The addition in $R[X; D]$ is the natural one and the multiplication is generated by the commutation rule $x_i a = a x_i + \delta_i(a)$, for $i \in I$. The ring $R[X; D]$ is called *a differential polynomial ring in several indeterminates.*

Proposition 1.3.13 ([2], Theorem 5.3)**.** *Let I be a non-empty countable set and let $S = R[X; D]$ be a differential polynomial ring in several indeterminates (as above). If S is left (right) quasi-duo, then $|I| = 1$.*

Next we present some examples of skew PBW extensions which are not quasi-duo.

Example 1.3.14. (a) The Weyl algebra $A_n(\mathbb{K}) = \mathbb{K}[t_1, \dots, t_n][x_1, \partial/\partial t_1] \cdots [x_n, \partial/\partial t_n]$ is an Ore extension. Note that, $x_i p = p x_i + \partial p/\partial t_i$, $x_i x_j - x_j x_i = 0$, for any $p \in \mathbb{K}[t_1, \dots, t_n]$ and $1 \leq i, j \leq n$. So, $A_n(\mathbb{K}) \simeq \sigma(\mathbb{K}[t_1, \dots, t_n])\langle x_1 \dots, x_n\rangle$ is a skew PBW extension. Since $A_n(\mathbb{K})$ is an Ore extension of derivation type, it is possible to apply Proposition 1.3.13, so $A_n(\mathbb{K})$ is not quasi-duo.

(b) Let $\mathbb{K}(t_1, \dots, t_n)$ the field of fractions of $\mathbb{K}[t_1, \dots, t_n]$, then extended Weyl algebra $B_n(\mathbb{K}) = \mathbb{K}(t_1, \dots, t_n)[x_1, \partial/\partial t_1] \cdots [x_n, \partial/\partial t_n]$ is also a skew PBW extension. Analogous to the previous example $B_n(\mathbb{K})$ is not quasi-duo.

CHAPTER 2

On the ACCP in skew PBW extensions

The results established in this chapter are the most important of this work, since all of them generalize those obtained by Nasr-Isfahani [31] for Ore extensions. This chapter is divided into two sections: in the first we will work on domains, and in the second, we will work on more general rings, the Σ-rigids rings.

2.1 ACCP over domains

In this section we establish necessary and sufficient conditions to guarantee that a skew PBW extension is an ACCP-domain. We start with the following caracterization of ACCPL domains.

Proposition 2.1.1 ([29], Proposition 2.7)**.** *For any domain B, the following conditions are equivalent:*

(i) *B satisfies ACCPL.*

(ii) *For any sequences $(a_m)_{m\in\mathbb{N}}$, $(b_m)_{m\in\mathbb{N}}$ of nonzero elements of B such that $a_m = b_m a_{m+1}$, for all $m \in \mathbb{N}$, there exists $s \in \mathbb{N}$ with $b_m \in U(B)$, for all $m \geq s$.*

(iii) *For any sequences $(a_m)_{m\in\mathbb{N}}$, $(b_m)_{m\in\mathbb{N}}$ of nonzero elements of B such that $a_m = b_m a_{m+1}$, for all $m \in \mathbb{N}$, there exists $s \in \mathbb{N}$ with $b_s \in U(B)$.*

(iv) *$\bigcap_{m\in\mathbb{N}} r_1 r_2 \cdots r_m B = 0$, for any sequence $(r_m)_{m\in\mathbb{N}}$ of nonunits of B.*

If C is a subring of a domain B such that $U(C) = C \cap U(B)$, where B satisfies ACCPL, then C also satisfies ACCPL ([29], Corollary 2.8). Next, we present the first important result of the chapter. Our Theorem 2.1.2 generalizes Nasr-Isfahani ([31], Theorem 2.3).

Theorem 2.1.2. *If A is a skew PBW extension of R, then the following assertions are equivalent:*

(1) *A is an ACCPL-domain;*

(2) $G(A)$ *is an ACCPL-domain.*

(3) R *is an ACCPL-domain.*

Proof. (1) $\Rightarrow$ (3) Suppose that A is an ACCPL-domain. Using that A is a domain, it is clear that R is a domain, and having in mind that $U(R) = R \cap U(A)$, R is an ACCPL-domain by Mazurek and Ziembowski ([29], Corollary 2.8).

(3) $\Rightarrow$ (1) Consider R an ACCPL-domain. By Proposition 1.1.11 we know that A is a domain. Let $(f_m)_{m\in\mathbb{N}}$, $(g_m)_{m\in\mathbb{N}}$ be sequences of nonzero elements of A with $f_m = g_m f_{m+1}$, for every $m \in \mathbb{N}$. Since A is a domain and σ_i is injective ($1 \leq i \leq n$), then $\deg(f_m) = \deg(g_m) + \deg(f_{m+1})$, for each m. Note that if for every $m \in \mathbb{N}$, $\deg(f_m) = \deg(f_{m+1})$, then $g_m \in R$ whence $\operatorname{lc}(f_m) = g_m \operatorname{lc}(f_{m+1})$. Since R is an ACCPL-domain, there exists $s \in \mathbb{N}$ such that $g_s \in U(R)$ (Proposition 2.1.1) which shows that A is an ACCPL-domain. Now, if there exists $m \in \mathbb{N}$ with $\deg(g_m) \neq 0$, then $\deg(f_m) > \deg(f_{m+1})$, and if, for each $s > m$, $\deg(g_s) = 0$, then by the same argument as above, there exists $m' > m$ such that $g_{m'} \in U(R)$ and the assertion follows. In this way, we assume that there exists a sequence of positive integers $m_1 < m_2 < m_3 < \cdots$ with $\deg(g_{m_i}) \neq 0$, for every integer i. Thus, $\deg(f_{m_1}) > \deg(f_{m_2}) > \deg(f_{m_3}) > \cdots$ and so there exists a positive integer t such that, for every $m \geq t$, $\deg(f_m) = 0$. Therefore, for each $m \geq t$, f_m, $g_m \in R$ and so there exists $m' > t$ with $g_{m'} \in U(R)$ which concludes the proof.

(2) $\Leftrightarrow$ (3) The proof of this equivalence uses similar arguments to the established in the proof of the equivalence (1) $\Leftrightarrow$ (3). $\square$

Corollary 2.1.3 ([31], Theorem 2.3)**.** *Let R be a ring, σ an endomorphism of the ring R and δ an σ-derivation of R. Then the following are equivalent:*

(1) $R[x; \sigma, \delta]$ *is an ACCPL-domain and* σ *is injective.*

(2) $R[[x; \sigma]]$ *is an ACCPL-domain.*

(3) $R[x; \sigma]$ *is an ACCPL-domain.*

(4) R *is an ACCPL-domain and* σ *is injective.*

Following Nasr-Isfahani [31], an endomorphism σ of a ring R *preserves nonunit elements of* R, if we have $\sigma(R \setminus U(R)) \subseteq R \setminus U(R)$. Thinking in our subject of interest, we will say that the family of injective endomorphisms $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ (Proposition 1.1.5) preserves nonunit elements of R, if every $\sigma_i \in \Sigma$ preserves nonunit elements of R. The next theorem extends Nasr-Isfahani ([31], Theorem 2.4).

Theorem 2.1.4. *Let A be a skew PBW extension of a ring R. If R is an ACCPR-domain and Σ preserves nonunit elements of R, then A is an ACCPR-domain.*

Proof. As we saw above, A is a domain, so consider $(f_m)_{m\in\mathbb{N}}$, $(g_m)_{m\in\mathbb{N}}$ sequences of nonzero elements of A with $f_m = f_{m+1} g_m$, for every $m \in \mathbb{N}$. Using that A is a domain and every $\sigma_i \in \Sigma$ is injective, $\deg(f_m) = \deg(f_{m+1}) + \deg(g_m)$, for each $m \in \mathbb{N}$. If, for $m \in \mathbb{N}$, $\deg(f_m) = \deg(f_{m+1}) = t$, we obtain that $g_m \in R$. Note that if, $f_{m+1} = a_0 + a_1 X_1 + \cdots + a_p X_p$, with $X_1 \prec X_2 \prec \cdots \prec X_p$, then $f_{m+1} g_m = a_0 g_m + a_1 X_1 g_m + \cdots +$

$a_pX_pg_m = a_0 + a_1X_1g_m + \cdots + a_p[\sigma^{\alpha_p}(g_m)X_p + p_{\alpha_p,g_m}]$ (by Proposition 1.1.10) whence $\mathrm{lc}(f_{m+1}g_m) = a_p\sigma^{\alpha_p}(g_m) = \mathrm{lc}(f_{m+1})\sigma^{\alpha_p}(g_m)$. Using that R is an ACCPR-domain, by the right-sided version of Proposition 2.1.1, there exists $m' \in \mathbb{N}$ with $\sigma^{\alpha_p}(g_{m'}) \in U(R)$, and having in mind that σ_i preserves nonunit elements of R, for every $1 \leq i \leq n$, $g_{m'} \in U(R)$, and so $g_{m'} \in U(A)$. In this way, if we assume that there exists $s \in \mathbb{N}$ with $\deg(g_s) \neq 0$, by a similar reasoning to the proof of Theorem 2.1.2, we obtain that $g_{m'} \in U(A)$, for some $m' \in \mathbb{N}$. Therefore, the right-sided version of Proposition 2.1.1 guarantees that A is an ACCPR-domain. □

Corollary 2.1.5 ([31], Theorem 2.4)**.** *Let R be a ring, σ an endomorphism of the ring R and δ an σ- derivation of R. If R is an ACCPR-domain and σ is injective and preserves nonunit elements of R, then $R[x;\sigma,\delta]$ is an ACCPR-domain.*

The next theorem 2.1.6 extends Nasr-Isfahani ([31], Theorem 2.5).

Theorem 2.1.6. *If A is a skew PBW extension of R, then $G(A)$ is an ACCPR-domain if and only if R is an ACCPR-domain and Σ preserves nonunit elements of R.*

Proof. Suppose that $G(A)$ is an ACCPR-domain. As we saw above, $U(R) = R \cap U(A)$, so the right-sided version of Mazurek and Ziembowski ([29], Corollary 2.8), implies that R is an ACCPR-domain. Now, if $\sigma_i(r) \in U(R)$, for some $r \in R \setminus U(R)$ and every $1 \leq i \leq n$, for each $m \in \mathbb{N}$ consider the element $f_m := (\sigma_i(r))^{-m}x_i$. In this way, $f_{m+1}r = (\sigma_i(r))^{-(m+1)}x_ir = (\sigma_i(r))^{-m-1}\sigma_i(r)x_i = f_m$, for every $m \in \mathbb{N}$. Hence, by the right-sided version of Proposition 2.1.1 we obtain that $r \in U(R)$ which contradicts our assumption. Therefore σ_i preserves nonunit elements of R, for every i, and so Σ preserves nonunit elements of R. The converse follows from Theorem 2.1.4. □

Corollary 2.1.7 ([31], Theorem 2.5)**.** *Let R be a ring and σ an endomorphism of the ring R. Then the following are equivalent:*

(1) *$R[x;\sigma]$ is an ACCPR-domain.*

(2) *$R[[x;\sigma]]$ is an ACCPR-domain.*

(3) *R is an ACCPR-domain and σ is injective and preserves nonunit elements of R.*

Proposition 1.1.12 and Theorem 2.1.6 imply the following result

Corollary 2.1.8. *Let A be a skew PBW extension of R. If every σ_i-derivation $\delta_i \in \Delta$ is inner, for $i = 1, \ldots, n$, then A is an ACCPR-domain if and only if R is an ACCPR-domain and Σ preserves nonunit elements of R.*

Definition 2.1.9. Let B be a domain. B is called to be *left (resp. right) Archimedian*, if we have $\bigcap_{m\geq 1} a^mB = 0$ ($\bigcap_{m\geq 1} Ba^m = 0$), for each nonunit element a of B.

Note that by Proposition 2.1.1, any ACCPL-domain (resp. ACCPR-domain) is left (resp. right) Archimedian, but the converse is not true in general (see Dumitrescu, [6] for a counterexample).

Theorems 2.1.10 and 2.1.12 generalize Nasr-Isfahani ([31], Theorems 2.8 and 2.9), respectively.

Theorem 2.1.10. *Let A be a skew PBW extension of R. If R is a right Archimedian domain and Σ preserves nonunit elements of R, then A is a right Archimedian domain.*

Proof. We know that A is a domain, so let us show that A is right Archimedian. Let $f \in A$ be a nonunit element and consider $g \in \bigcap_{m\geq 1} Af^m$. It is clear that for every $m \in \mathbb{N}$ there exists an element $h_m \in A$, $h_m = a_{m,0} + a_1 X_{m,1} + \cdots + a_p X_{m,p}$, say, with $g = h_m f^m$. Consider the following two cases: (i) if $\deg(f) = 0$, then $\text{lc}(g) = \text{lc}(h_m)\sigma^{\alpha_{m,p}}(\text{lc}(f^m))$ which shows that $\text{lc}(g) \in \bigcap_{m\geq 1} R(\sigma^{\alpha_{m,p}}(f))^m$, but having in mind that $\sigma^{\alpha_{m,p}}(f)$ is a nonunit element, necessarily $\text{lc}(g) = 0$ whence $g = 0$. (ii) if $\deg(f) \neq 0$, then for every $m \in \mathbb{N}$ we know that $\deg(g) = \deg(h_m) + m\deg(f)$ which shows that $g = 0$. □

Corollary 2.1.11 ([31], Theorem 2.8)**.** *Let R be a ring, σ an endomorphism of the ring R and δ an σ- derivation of R. If R is a right archimedean domain and σ is injective and preserves nonunit elements of R, then $R[x;\sigma,\delta]$ is a right archimedean domain.*

Theorem 2.1.12. *If A is a skew PBW extension of R, then $G(A)$ is a right Archimedian domain if and only if R is a right Archimedian domain and Σ preserves nonunit elements of R.*

Proof. Suppose that $G(A)$ is a right Archimedian domain. Since $R \subseteq G(A)$, it is clear that R is a domain. Consider $a \in R$, a nonunit element, and let $b \in \bigcap_{m\geq 1} Ra^m$. We can see that $b \in \bigcap_{m\geq 1} G(A)a^m$, and so $b = 0$, which shows that R is a right Archimedian domain. Now, let $\sigma_i(r) \in U(R)$, for some nonunit r of R. If $m \in \mathbb{N}$ then we consider the element $f_m := (\sigma_i(r))^{-m}x_i$ of $G(A)$, which satisfies $f_m r^m = x_i$, that is, $x_i \in \bigcap_{m\geq 1} G(A)r^m$ which is clearly a contradiction, so σ_i preservers nonunit elements of R, for every $1 \leq i \leq n$. The converse of the assertion follows from Theorem 2.1.10. □

Corollary 2.1.13 ([31], Theorem 2.9)**.** *Let R be a ring and σ an endomorphism of the ring R. Then the following are equivalent:*

(1) *$R[x;\sigma]$ is a right archimedean domain.*

(2) *$R[[x;\sigma]]$ is a right archimedean domain.*

(3) *R is a right archimedean domain and σ is injective and preserves nonunit elements of R.*

Proposition 1.1.12 and Theorem 2.1.12 guarantee the following corollary.

Corollary 2.1.14. *Let A be a skew PBW extension of R. If every σ_i-derivation $\delta_i \in \Delta$ is inner, for $i = 1, \ldots, n$ then A is a right Archimedian domain if and only if R is a right Archimedian domain and Σ preservers nonunit elements of R.*

The last result of this section extends Nasr-Isfahani ([31], Theorem 2.11).

Theorem 2.1.15. *If A is a skew PBW extension of R, then A is left Archimedian domain if and only if $G(A)$ is a left Archimedian domain if and only if R is a left Archimedian domain.*

Proof. Suppose that $G(A)$ is a left Archimedian domain. Since $R \subseteq G(A)$, it is clear that R is a domain. Consider $a \in R$, a nonunit element, and let $b \in \bigcap_{m\geq 1} a^m R$. We can see that $b \in \bigcap_{m\geq 1} a^m G(A)$, and so $b = 0$, which shows that R is a left Archimedian domain. Now, suppose that R is a left Archimedian domain. We know that A is a domain, so let us show that A is left Archimedian. Let $f \in A$ be a nonunit element and consider $g \in \bigcap_{m\geq 1} f^m A$. It is clear that for every $m \in \mathbb{N}$ there exists an element $h_m \in A$, $h_m = a_{m,0} + a_1 X_{m,1} + \cdots + a_p X_{m,p}$, say, with $g = f^m h_m$. Consider the following two cases: (i) if $\deg(f) = 0$, then $\mathrm{lc}(g) = \mathrm{lc}(f^m)\mathrm{lc}(h_m)$ which shows that $\mathrm{lc}(g) \in \bigcap_{m\geq 1} f^m R$, but having in mind that f is a nonunit element, necessarily $\mathrm{lc}(g) = 0$ whence $g = 0$. (ii) if $\deg(f) \neq 0$, then for every $m \in \mathbb{N}$ we know that $\deg(g) = \deg(h_m) + m\deg(f)$ which shows that $g = 0$. Finally, suppose that A is a left Archimedian domain. Using that A is a domain, it is clear that R is a domain. Consider $a \in R$, a nonunit element, and let $b \in \bigcap_{m\geq 1} a^m R$. We can see that $b \in \bigcap_{m\geq 1} a^m A$, and so $b = 0$, which shows that R is a left Archimedian domain. □

Corollary 2.1.16 ([31], Theorem 2.11)**.** *Let R be a ring, σ an endomorphism of the ring R and δ an σ-derivation of R. Then the following are equivalent:*

(1) *$R[x;\sigma,\delta]$ is a left archimedean domain and σ is injective.*

(2) *$R[[x;\sigma]]$ is a left archimedean domain.*

(3) *$R[x;\sigma]$ is a left archimedean domain.*

(4) *R is a left archimedean domain and σ is injective.*

2.2 ACCP over Σ-rigid rings

Frohn ([8], Theorem 4.1) proved that if a commutative ring R satisfies ACCP and the commutative polynomial ring $R[x]$ has ascending chain condition on annihilator ideals, then $R[x]$ also satisfies ACCP. The purpose in this section is to extend this result to the context of skew PBW extensions and hence to generalize the results presented in Nasr-Isfahani ([31], Section 3) for Ore extensions.

Considering the Ore extension $R[x;\sigma,\delta]$, Krempa [17] defined σ as a *rigid endomorphism*, if $r\sigma(r) = 0$ implies $r = 0$, for $r \in R$. Krempa called R σ-rigid if there exists a rigid endomorphism σ of R. Note that σ-rigid rings are reduced, i.e., has no nonzero nilpotent elements. An important fact about these rings is that if a reduced ring R satisfies the ascending chain condition on right annihilators, then R satisfies the ascending chain condition on left annihilators (this remark will be important in the proof of Theorem 2.2.5).

Since Ore extensions of injective type are particular examples of skew PBW extensions, we recall the following definition with the purpose of studying the notion of *rigidness* for these extensions.

Definition 2.2.1 ([38], Definition 3.2)**.** Let R be a ring and Σ a family of endomorphisms of R. Σ is called a *rigid endomorphisms family*, if $r\sigma^{\alpha}(r) = 0$ implies $r = 0$, for every

$r \in R$ and $\alpha \in \mathbb{N}^n$. A ring R is called to be Σ-*rigid*, if there exists a rigid endomorphisms family Σ of R.

Note that if Σ is a rigid endomorphisms family, then every element $\sigma_i \in \Sigma$ is a monomorphism. In fact, Σ-rigid rings are reduced rings: if R is a Σ-rigid ring and $r^2 = 0$ for $r \in R$, then we have the equalities $0 = r\sigma^{\alpha}(r^2)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r\sigma^{\alpha}(r))$, i.e., $r\sigma^{\alpha}(r) = 0$ and so $r = 0$, that is, R is reduced (note that there exists an endomorphism of a reduced ring which is not a rigid endomorphism, see Hong, [14], Example 9). With this in mind, we consider the family of injective endomorphisms Σ and the family Δ of Σ-derivations in a skew PBW extension A of a ring R (see Proposition 1.1.5). As a matter of fact, the notion of rigidness was very useful Reyes [38] for the study of Baer, quasi-Baer, p.p. and p.q. Baer rings over skew PBW extensions (see also Niño and Reyes [32], Reyes and Suárez [38], [39], [41] and [43] for related properties with the notion of rigid ring over skew PBW extensions).

Example 2.2.2. We present remarkable examples of skew PBW extensions over Σ-rigid rings (see Reyes [35], Lezama and Reyes [24], and Reyes and Suárez [43] for a detailed definition and reference of every example).

(a) If A is a constant skew PBW extension, then it is clear that R is a Σ-rigid ring.

(b) We also encounter examples of skew PBW extensions which are not constant over Σ-rigid rings: (i) the quantum plane $\mathcal{O}_q(\mathbb{k}^2)$; the algebra of q-differential operators $D_{q,h}[x,y]$; the mixed algebra D_h; the operator differential rings; the algebra of differential operators $D_{\mathbf{q}}(S_{\mathbf{q}})$ on a quantum space $S_{\mathbf{q}}$, and more.

(c) It is important to say that several algebras of quantum physics can be expressed as skew PBW extensions (for instance, Weyl algebras, additive and multiplicative analogue of the Weyl algebra, quantum Weyl algebras, q-Heisenberg algebra, and others), which allows us to characterize several properties with physical meaning. As Curado [5] say, "algebraic methods have long been applied to the solution of a large number of quantum physical systems. In the last decades, quantum algebras appeared in the framework of quantum integrable one-dimensional models and have ever since been applied to many physical phenomena [...] It was found that it could be generalized leading to the concept of deformed Heisenberg algebras that have been used in many areas, as nuclear physics, condensed matter, atomic physics, etc". With these ideas in mind, next we present some remarkable examples of these algebras (the proof that these algebras are skew PBW extensions can be realized using the theory developed in Reyes and Suárez [40]) over Σ-rigid rings.

   (i) The Lie-deformed Heisenberg algebra introduced by Jannussis is defined by the commutation relations

$$q_j(1+i\lambda_{jk})p_k - p_k(1-i\lambda_{jk})q_j = i\hbar\delta_{jk}$$
$$[q_j, q_k] = [p_j, p_k] = 0, \quad j,k = 1,2,3,$$

   where q_j, p_j are the position and momentum operators, and $\lambda_{jk} = \lambda_k\delta_{jk}$, with λ_k real parameters. If $\lambda_{jk} = 0$ one recovers the usual Heisenberg algebra.

(ii) The *quantum Weyl algebra* introduced by Giaquinto and Zhang with the aim of studying the Jordan Hecke symmetry is as a quantization of the usual second Weyl algebra. By definition, $A_2(J_{a,b})$ is the $\Bbbk$-algebra generated by the variables $x_1, x_2, \partial_1, \partial_2$, with relations (depending on parameters $a, b \in \Bbbk$)

$$\begin{aligned} x_1x_2 &= x_2x_1 + ax_1^2, & \partial_2\partial_1 &= \partial_1\partial_2 + b\partial_2^2 \\ \partial_1x_1 &= 1 + x_1\partial_1 + ax_1\partial_2, & \partial_1x_2 &= -ax_1\partial_1 - abx_1\partial_2 + x_2\partial_1 + bx_2\partial_2 \\ \partial_2x_1 &= x_1\partial_2, & \partial_2x_2 &= 1 - bx_1\partial_2 + x_2\partial_2. \end{aligned}$$

Over any field $\Bbbk$, if $a = b = 0$, then $A_2(J_{0,0}) \simeq A_2$, the usual second Weyl algebra.

(iii) With the purpose of obtaining bosonic representations of the Drinfield-Jimbo quantum algebras, Hayashi considered the algebra $\mathbf{U}$. Let us see its construction. Let $\mathbf{U}$ be the algebra generated by the indeterminates $\omega_1, \dots, \omega_n, \psi_1, \dots, \psi_n$, $\psi_1^*, \dots, \psi_n^*$, with the relations

$$\begin{aligned} \psi_j\psi_i - \psi_i\psi_j = \psi_j^*\psi_i^* - \psi_i^*\psi_j^* = \omega_j\omega_i - \omega_i\omega_j = \psi_j^*\psi_i - \psi_i\psi_j^* &= 0, \quad 1 \leq i < j \leq n, \\ \omega_j\psi_i - q^{-\delta_{ij}}\psi_i\omega_j = \psi_j^*\omega_i - q^{-\delta_{ij}}\omega_i\psi_j^* &= 0, \quad 1 \leq i, j \leq n, \\ \psi_i^*\psi_i - q^2\psi_i\psi_i^* = -q^2\omega_i^2, \quad q \in \mathbb{C} & \quad 1 \leq i \leq n. \end{aligned}$$

(iv) Jannussis, studied the non-Hermitian realization of a Lie deformed, a non-canonical Heisenberg algebra, considering the case of operators A_j, B_k which are non-Hermitian (i.e., $\hbar = 1$)

$$\begin{aligned} A_j(1 + i\lambda_{jk})B_k - B_k(1 - i\lambda_{jk})A_j &= i\delta_{jk} \\ [A_j, B_k] &= 0 \quad (j \neq k) \\ [A_j, A_k] &= [B_j, B_k] = 0, \end{aligned}$$

and,

$$\begin{aligned} A_j^+(1 + i\lambda_{jk})B_k^+ - B_k^+(1 - i\lambda_{jk})_j^+ &= i\delta_{jk} \\ [A_j^+, B_k^+] &= 0 \ (j \neq k), \\ [A_j^+, A_k^+] &= [B_j^+, B_k^+] = 0 \end{aligned}$$

where $A_j \neq A_j^+$, $B_k \neq B_k^+$ $(j, k = 1, 2, 3)$. If the operators A_j, B_k are in the form $A_j = f_j(N_j{+}1)a_j$, $B_k = a_k^+ f_k(N_k{+}1)$, where a_j, a_j^+ are leader operators of the usual Heisenberg-Weyl algebra, with N_j the corresponding number operator ($N_j = a_j^+ a_j$, $N_j \mid n_j\rangle = n_j|n_j\rangle$), and the structure functions $f_j(N_j{+}1)$ complex, then it is showed that A_j and B_k are given by

$$\begin{aligned} A_j &= \sqrt{\frac{i}{1 + i\lambda_j}} \left(\frac{[(1 - i\lambda_j)/(1 + i\lambda_j)]^{N_j+1} - 1}{(1 - i\lambda_j)/(1 + i\lambda_j) - 1} \frac{1}{N_j + 1} \right)^{\frac{1}{2}} a_j \\ B_k &= \sqrt{\frac{i}{1 + i\lambda_k}} a_k^+ \left(\frac{[(1 - i\lambda_k)/(1 + i\lambda_k)]^{N_k+1} - 1}{(1 - i\lambda_k)/(1 + i\lambda_k) - 1} \frac{1}{N_k + 1} \right)^{\frac{1}{2}}. \end{aligned}$$

Recall that if A is a skew PBW extension of R where the elements $c_{i,j}$ are invertible in R, then R is Σ-rigid if and only if A is a reduced ring (Reyes [38], Proposition 3.5). The next proposition establish some useful results about Σ-rigid rings.

Proposition 2.2.3 ([38], Lemma 3.3). *If R is a Σ-rigid ring and a, b are elements of R, then:*

(i) *If $ab = 0$, then $a\sigma^{\alpha}(b) = \sigma^{\alpha}(a)b = 0$, for every $\alpha \in \mathbb{N}^n$.*

(ii) *If $ab = 0$, then $a\delta^{\beta}(b) = \delta^{\beta}(a)b = 0$, for each element $\beta \in \mathbb{N}^n$.*

(iii) *If $ab = 0$, then $a\sigma^{\alpha}(\delta^{\beta}(b)) = a\delta^{\beta}(\sigma^{\alpha}(b)) = 0$, for every $\alpha,\ \beta \in \mathbb{N}^n$.*

(iv) *If $a\sigma^{\theta}(b) = 0$, for some element $\theta \in \mathbb{N}^n$, then $ab = 0$.*

Note that if I is an ideal of a ring R, where R is an ACCPL (resp. ACCPR) ring, then R/I is an ACCPL (resp. ACCPR) ring ([31], Lemma 3.3).

For the next theorem, Theorem 2.2.5, we need some preliminary facts and a proposition (Proposition 2.2.4) about quotients of skew PBW extensions: consider $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ a skew PBW extension of a ring R. Let $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$ such as in Proposition 1.1.5. Following Reyes [37] (see also Lezama, [23]), if I is an ideal of R, I is called Σ-invariant (Δ-invariant), if it is invariant under each injective endomorphism σ_i (σ_i-derivation δ_i) of Σ (Δ), that is, $\sigma_i(I) \subseteq I$ ($\delta_i(I) \subseteq I$), for $1 \le i \le n$. If I is both Σ and Δ-invariant ideal we say that I is (Σ, Δ)-invariant.

Proposition 2.2.4 ([23], Proposition 2.6; [37], Proposition 4.1). *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R and I is a (Σ, Δ)-invariant ideal of R, then the following statements hold:*

(i) *IA is an ideal of A and $IA \cap R = I$. IA is a proper ideal of A if and only if I is proper in R. Moreover, if σ_i is bijective and $\sigma_i(I) = I$, for every i, then $IA = AI$.*

(ii) *If I is proper and $\sigma_i(I) = I$, for every $1 \le i \le n$, then A/IA is a skew PBW extension of R/I. In fact, if I is proper and A is bijective, then A/IA is a bijective skew PBW extension of R/I.*

Proof. We follow the ideas presented in [23].

(i) It is clear that IA is a right ideal, but since I is (Σ, Δ)-invariant, then IA is also a left ideal of A. It is obvious that $IA \cap R = I$. From this last equality we get also that IA is proper if and only if I is proper. Using again that I is (Σ, Δ)-invariant, we get that $AI \subseteq IA$. Assuming that σ_i is bijective and $\sigma_i(I) = I$ for every i, then $IA \subseteq AI$.

(ii) According to (i), we only have to show that $\overline{A} := A/IA$ is a skew PBW extension of $\overline{R} := R/I$. For this we will verify the four conditions of Definition 1.1.4. It is clear that $\overline{R} \subseteq \overline{A}$. Moreover, $\overline{A}$ is a left $\overline{R}$-module with generating set $\text{Mon}\{\overline{x_1}, \dots, \overline{x_n}\}$. Next we show that $\text{Mon}\{\overline{x_1}, \dots, \overline{x_n}\}$ is independent. Consider the expression $\overline{r_1}\overline{X_1} + \cdots + \overline{r_n}\overline{X_n} = 0$, where $X_i \in \text{Mon(A)}$ for each i. We have $r_1X_1 + \cdots + r_nX_n \in IA$ and hence

$$r_1X_1 + \cdots + r_nX_n = r_1'X_1 + \cdots + r_n'X_n, \text{ for some } r_i' \in I, i = 1, \dots, n.$$

Thus, $(r_1 - r'_1)X_1 + \cdots + (r_n - r'_n)X_n = 0$, so $r_i \in I$, i.e., $\overline{r_i} = \overline{0}$ for $i = 1, \ldots, n$. Let $\overline{r} \neq \overline{0}$ with $r \in R$. Then $r \notin IA$, and hence, $r \notin I$, in particular, $r \neq 0$ and there exists $c_{i,r} := \sigma_i(r) \neq 0$ such that $x_i r = c_{i,r}x_i + \delta_i(r)$. Thus, $\overline{x_i}\overline{r} = \overline{c}_{i,r}\overline{x}_i + \overline{\delta_i(r)}$. Observe that $c_{i,r} \neq 0$, contrary $c_{i,r} = \sigma_i(r) \in IA \cap R = I = \sigma_i(I)$, i.e, $r \in I$, a contradiction. This completes the proof of condition (iii) in Definition 1.1.4. In A we have $x_jx_i - c_{i,j}x_ix_j \in R + \sum_{t=1}^{n} Rx_t$, with $c_{i,j} \in R - \{0\}$, so in $\overline{A}$ we get that $\overline{x}_j\overline{x}_i - \overline{c}_{i,j}\overline{x}_i\overline{x}_j \in \overline{R} + \sum_{t=1}^{n} \overline{R}\overline{x_t}$. Since I is proper and $c_{i,j}$ is left invertible for $i < j$ and right invertible for $i > j$, then $\overline{c_{i,j}} \neq \overline{0}$. This completes the proof of condition (iv) in Definition 1.1.4. If σ_i is bijective, then $\overline{\sigma_i}(\overline{r}) := \overline{\sigma_i(r)}$ is bijective.

□

From Proposition 2.2.4, we can see that if I is (Σ, Δ)-invariant, then over $\overline{R} := R/I$ it is induced a system $(\overline{\Sigma}, \overline{\Delta})$ of endomorphisms $\overline{\Sigma}$ and $\overline{\Sigma}$-derivations $\overline{\Delta}$, defined by $\overline{\sigma_i}(\overline{r}) = \overline{\sigma_i(r)}$ and $\overline{\delta_i}(\overline{r}) = \overline{\delta_i(r)}$, for $1 \leq i \leq n$. We keep the variables $x_1, \ldots, x_n$ of extension A to the extension A/IA if no confusion arises.

Our next theorem extends Nasr-Isfahani ([31], Theorem 3.4). We use similar arguments to the established by Frohn ([8], Theorem 4.1).

Theorem 2.2.5. *Let A be a bijective skew PBW extension of a Σ-rigid and ACCPR ring R. If R satisfies the ACC on right annihilators, then A is an ACCPR ring.*

Proof. Let $f \in A$ and consider the set I_f which consists of the leading coefficients of elements of the ideal AfA, including the zero element. One can see that I_f is an ideal of R. With the aim of proving the theorem, we will assume that there exists at least a nonstabilizing chain of principal right ideals of A. With this in mind, we consider the set

$$M := \{l_R(\bigcup_{i\geq 1} I_{g_i}) \mid g_1A \subseteq g_2A \subseteq \cdots\},$$

where $g_1A \subseteq g_2A \subseteq \cdots$ is a nonstabilizing chain of principal right ideals. By the rigidness of R, we know that R is reduced and since R satisfies the ascending chain condition on right annihilators, then R satisfies the ascending chain condition on left annihilators. This fact allows us to guarantee that M has a maximal element P, say, where $P := l_R(\bigcup_{i\geq 1} I_{f_i})$, with $f_1A \subseteq f_2A \subseteq \cdots$ a nonstabilizing chain of A. The idea is to prove that P is a completely prime ideal of R. If this is not the case, then for two elements $a, b \in R$, we have $ab \in P$ and $a, b \notin P$. For every element f_i, we consider the polynomial bf_i, and using Proposition 2.2.3 together with the fact that $ab \in P$, we can see that $a \in l_R(\bigcup_{i\geq 1} I_{bf_i})$. Note also that $P \subseteq l_R(\bigcup_{i\geq 1} I_{bf_i})$ which means that the chain $bf_1A \subseteq bf_2A \subseteq \cdots$ stabilizes (since P is maximal), that is, there exists $t \in \mathbb{N}$ such that $bf_{m+1} = bf_mh_m$, for every $m \geq t$ and some element $h_m \in A$. For every m, there exists $g_m \in A$ with $f_m = f_{m+1}g_m$, and hence $bf_{m+1}(1 - g_mh_m) = 0$. Define $q_i = f_i(1 - g_{i-1}h_{i-1})$, for $i > t$. Using that R is reduced, we can prove that $b \in l_R(\bigcup_i I_{q_i})$ and $P \subseteq l_R(\bigcup_i I_{q_i})$. Therefore, the chain $q_1A \subseteq q_2A \subseteq \cdots$ stabilizes. Let $t' \in \mathbb{N}$ such that for every $s \geq t'$, $q_{s+1} = q_sl_s$, for some $l_s \in A$. Note that $f_{s+1}(1 - g_sh_s) = f_s(1 - g_{s-1}h_{s-1})l_s$, that is, $f_{s+1} = f_sh_s + f_s(1 - g_{s-1}h_{s-1})l_s$ whence $f_{s+1} \in f_sA$ which is a contradiction. This argument proves that P is completely prime.

Now, since R is Σ-rigid and $P = l_R(\bigcup_{i\geq 1} I_{f_i})$, using Proposition 2.2.3 we can see that $\sigma_i(P) = P$ and $\delta_i(P) \subseteq P$, for $i = 1, \ldots, n$. This allows us to consider the skew PBW extension $\sigma(R/P)\langle x_1, \ldots, x_n\rangle$ with family of injective endomorphisms $\overline{\Sigma}$ and $\overline{\Sigma}$-derivations $\overline{\Delta}$ as given above. Since R is ACCPR and P is a completely prime ideal of R, then R/P is an ACCPR-domain ([31], Lemma 3.3), and hence Theorem 2.1.4 establishes that $\sigma(R/P)\langle x_1, \ldots, x_n\rangle$ is an ACCPR-domain (note that σ_i is bijective by assumption, and $\sigma_i(P) = P$, for each i, so Proposition 2.2.4 establishes that $\overline{\sigma_i}$ is bijective, and hence $\overline{\sigma_i}$ preserves nonunit elements of R/P, for $i = 1, \ldots, n$). Now, for every positive integer i, let $\overline{f_i} = \overline{f_{i+1}}\overline{g_i}$, where we consider the expression $\overline{f} = (a_0+P)+(a_1+P)X_1+\cdots+(a_q+P)X_q \in \sigma(R/P)\langle x_1, \ldots, x_n\rangle$, for the element $f = a_0 + a_1X_1 + \cdots + a_qX_q \in A$, with $\mathrm{lc}(f) = a_q \neq 0$ and $X_1 \prec X_2 \prec \cdots \prec X_q$. In the case that $\overline{f_i}$ is equal to zero, for some i, the leading coefficient $\mathrm{lc}(f)$ of f is an element of $P = l_R(\bigcup_{i\geq 1} I_{f_i})$, so $(\mathrm{lc}(f))^2 = 0$ whence $\mathrm{lc}(f) = 0$ (R is reduced) which is a contradiction. This fact guarantees that, for every i, $\overline{f_i} \neq 0$ and so $\overline{g_i} \neq 0$. From Proposition 2.1.1, we know that there exists a positive integer s' with $\overline{g_m}$ invertible in $\sigma(R/P)\langle x_1, \ldots, x_n\rangle$, for $m \geq s'$. Hence, there is an element $\overline{h} \in \sigma(R/P)\langle x_1, \ldots, x_n\rangle$ such that $\overline{g_m}\overline{h} = \overline{h}\overline{g_m} = \overline{1}$, i.e., $\overline{g_m}\overline{h} - \overline{1} = 0$ which means that for every coefficient b of the element $g_mh - 1$, it holds that $b \in P$. Next, we will show that $f_{m+1}(g_mh-1) = 0$. Consider the expression for f_{m+1} given by $f_{m+1} = a_0+a_1X_1+\cdots+a_lX_l$. It is clear that for each coefficient b of g_mh-1, $ba_l = 0$, and using that R is reduced, $a_lb = 0$, so Proposition 1.1.13, Remark 1.1.14 and Proposition 2.2.3 imply that $a_lX_lb = 0$, whence $f_{m+1}b = (a_0+a_1X_1+\cdots+a_{l-1}X_{l-1})b$, but having in mind that $f_{m+1}b \in Af_{m+1}A$, it follows that $a_{l-1}\sigma^{\alpha_{l-1}}(b)$ is an element of $I_{f_{m+1}}$ ($\sigma^{\alpha_{l-1}} = \exp(X_l)$). Then $ba_{l-1}\sigma^{\alpha_{l-1}}(b) = 0$ and since R is reduced, $a_{l-1}\sigma^{\alpha_{l-1}}(b)b = 0$. From Proposition 2.2.3 (iv), we have $a_{l-1}b^2 = 0$ and so $a_{l-1}b = 0$ (R is reduced). Thus, $a_{l-1}X_{l-1}b = 0$. If we proceed in this way, we have $a_iX_ib = 0$, for every $0 \leq i \leq l$, and hence $f_{m+1}b = 0$, i.e., $f_{m+1}(g_mh - 1) = 0$ whence $f_{m+1} = f_{m+1}g_mh = f_mh$. In this way, the chain $f_1A \subseteq f_2A \subseteq \cdots$ stabilizes, which contradicts our assumption. □

Corollary 2.2.6 ([31], Theorem 3.4). *Let R be an ACCPR ring, σ a rigid automorphism of R and δ an σ-derivation of R. If R satisfies the ACC on right annihilators, then $R[x;\sigma,\delta]$ is an ACCPR ring.*

For the context of Ore extensions, Nasr-Isfahani ([31], Example 3.5) presented an example which illustrates that the condition on rigidness on R can not be eliminated from the assumptions of the theorem.

With the aim of establishing the last result of the chapter, the Proposition 2.2.9, which generalizes Nasr-Isfahani ([31], Corollary 3.6), we state the Proposition 2.2.7 about the opposite ring of a skew PBW extension. Briefly, recall that the *opposite* of a ring is the ring with the same elements and addition operation, but with the multiplication performed in the reverse order. More precisely, the opposite of a ring $(R, +, \cdot)$ is the ring $(R^{\mathrm{op}}, +, *)$, whose multiplication $*$ is defined by $a * b = b \cdot a$.

Proposition 2.2.7 ([42], Proposition 4.1). *If A is a bijective skew PBW extension over R, then A^{op} is a bijective skew PBW extension over R^{op}. In fact, for A^{op} we have the automorphisms $\sigma_i^{\mathrm{op}} : R^{\mathrm{op}} \to R^{\mathrm{op}}$ given by $\sigma_i^{\mathrm{op}}(r) := \sigma_i^{-1}(r)$, and the σ_i^{op}-derivations $\delta_i^{\mathrm{op}} : R^{\mathrm{op}} \to R^{\mathrm{op}}$ defined by $\delta_i^{\mathrm{op}}(r) := -\delta_i(\sigma_i^{-1}(r))$, for every element $r \in R^{\mathrm{op}}$.*

Proof. Let $A = \sigma(R)\langle x_1 \dots, x_n\rangle$ be a bijective skew PBW extension of R. We will verify the four conditions of the Definition 1.1.4 for the rings R^{op} and A^{op}.

(i) It is clear that $R^{\text{op}} \subseteq A^{\text{op}}$.

(ii) Since A is a left free R-module with basis the set of monomials $\text{Mon}(A) := \{x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n\}$, then by the definition of the product in A^{op}, we have that A^{op} is a free right R-module with basis the set $\text{Mon}(A^{\text{op}}) := \{x^{\alpha^{\text{op}}} = x_n^{\alpha_n}\cdots x_1^{\alpha_1} \mid \alpha^{\text{op}} = (\alpha_n,\dots,\alpha_1) \in \mathbb{N}^n\}$. Hence, A^{op} is a left free R^{op}-module.

(iii) We will see that for each $1 \le i \le n$, and for every $r \in R^{\text{op}} \setminus \{0\}$, there exists $c'_{i,r} \in R^{\text{op}} \setminus \{0\}$ such that $rx_i - x_ic'_{i,r} \in R^{\text{op}}$. Put $c'_{i,r} := \sigma_i^{-1}(r)$. Given that

$$x_ic'_{i,r} = x_i\sigma_i^{-1}(r) = \sigma_i(\sigma_i^{-1}(r))x_i + \delta_i(\sigma_i^{-1}(r)) = rx_i + \delta_i(\sigma_i^{-1}(r)),$$

we have that $rx_i - x_ic'_{i,r} = -\delta_i(\sigma_i^{-1}(r)) \in R^{\text{op}}$.

(iv) Let $c'_{i,j} := \sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1}))$. Then

$$\begin{aligned}
x_ix_j - x_jx_ic'_{i,j} &= x_ix_j - x_jx_i\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1}))\\
&= x_ix_j - x_j[\sigma_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))x_i + \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))]\\
&= x_ix_j - x_j\sigma_j^{-1}(c_{i,j}^{-1})x_i - x_j\delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))\\
&= x_ix_j - [\sigma_j(\sigma_j^{-1}(c_{i,j}^{-1}))x_j + \delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))]x_i\\
&\quad - x_j\delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))\\
&= x_ix_j - c_{i,j}^{-1}x_jx_i - \delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))x_i - x_j\delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1}))) \qquad (2.2.1)
\end{aligned}$$

From Definition 1.1.4 (iv), we have that $x_jx_i - c_{i,j}x_ix_j = r^{(i,j)} + \sum_{l=1}^n r_l^{(i,j)}x_l$, whence $c_{i,j}^{-1}x_jx_i = x_ix_j + c_{i,j}^{-1}r^{(i,j)} + \sum_{l=1}^n c_{i,j}^{-1}r_l^{(i,j)}x_l$. So, by replacing the term $c_{i,j}^{-1}x_jx_i$ in the above expression (2.2.1), we have that

$$\begin{aligned}
x_ix_j - x_jx_ic'_{i,j} = &\ -c_{i,j}^{-1}r^{(i,j)} - \left(\sum_{l=1}^n c_{i,j}^{-1}r_l^{(i,j)}x_l\right)\\
&- \delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))x_i - x_j\delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))\\
= &\ -c_{i,j}^{-1}r^{(i,j)} - \left(\sum_{l=1}^n x_l\sigma_l^{-1}(c_{i,j}^{-1}r_l^{(i,j)}) - \delta_l(\sigma_l^{-1}(c_{i,j}^{-1}r_l^{(i,j)}))\right)\\
&- [x_i\sigma_i^{-1}(\delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))) - \delta_i(\sigma_i^{-1}(\delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))))]\\
&- x_j\delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))
\end{aligned}$$

or equivalently,

$$\begin{aligned}
x_ix_j - x_jx_ic'_{i,j} = &\ -c_{i,j}^{-1}r^{(i,j)} + \left(\sum_{l=1}^n \delta_l(\sigma_l^{-1}(c_{i,j}^{-1}r_l^{(i,j)}))\right)\\
&+ \delta_i(\sigma_i^{-1}(\delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))))\\
&- \sum_{l=1,\ l\neq i,j} x_l\sigma_l^{-1}(c_{i,j}^{-1}r_l^{(i,j)}) - x_i[\sigma_i^{-1}(c_{i,j}^{-1}r_i^{(i,j)})\\
&+ \sigma_i^{-1}(\delta_j(\sigma_j^{-1}(c_{i,j}^{-1})))] - x_j[\sigma_j^{-1}(c_{i,j}^{-1}r_j^{(i,j)}) + \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))],
\end{aligned}$$

which shows that $x_ix_j - x_jx_ic'_{i,j} \in R + x_1R + \cdots + x_nR$.

Finally, let r and r' be elements of R^{op}. We have:

$$\begin{aligned}\sigma_i^{\text{op}}(r+r') &= \sigma_i^{-1}(r+r') = \sigma_i^{-1}(r) + \sigma_i^{-1}(r') = \sigma_i^{\text{op}}(r) + \sigma_i^{\text{op}}(r')\\ \sigma_i^{\text{op}}(1_{R^{\text{op}}}) &= \sigma_i^{\text{op}}(1_R) = \sigma_i^{-1}(1_R) = 1_R = 1_{R^{\text{op}}}\\ \sigma_i^{\text{op}}(rr') &= \sigma_i^{-1}(r'r) = \sigma_i^{-1}(r')\sigma_i^{-1}(r) = \sigma_i^{-1}(r)\sigma_i^{-1}(r') = \sigma_i^{\text{op}}(r)\sigma_i^{\text{op}}(r').\end{aligned}$$

Given that σ_i is injective and surjective, so it is σ_i^{op}, for every $1 \le i \le n$. With respect to the functions δ_i^{op}, we have

$$\begin{aligned}\delta_i^{\text{op}}(r+r') &= -\delta_i(\sigma_i^{-1}(r+r')) = -\delta_i(\sigma_i^{-1}(r) + \sigma_i^{-1}(r'))\\ &= -\delta_i(\sigma_i^{-1}(r)) - \delta_i(\sigma_i^{-1}(r'))\\ &= \delta_i^{\text{op}}(r) + \delta_i^{\text{op}}(r'),\end{aligned}$$

and using the product on R^{op},

$$\begin{aligned}\delta_i^{\text{op}}(rr') &= -\delta_i(\sigma_i^{-1}(r'r)) = -\delta_i(\sigma_i^{-1}(r')\sigma_i^{-1}(r))\\ &= -[\sigma_i(\sigma_i^{-1}(r'))\delta_i(\sigma_i^{-1}(r')) + \delta_i(\sigma_i^{-1}(r'))\sigma_i^{-1}(r)]\\ &= -r'\delta_i(\sigma_i^{-1}(r')) - \delta_i(\sigma_i^{-1}(r'))\sigma_i^{-1}(r)\\ &= \sigma_i^{-1}(r)(-\delta_i(\sigma_i^{-1}(r'))) + (-\delta_i(\sigma_i^{-1}(r)))r'\\ &= \sigma_i^{\text{op}}(r)\delta_i^{\text{op}}(r') + \delta_i^{\text{op}}(r)r',\end{aligned}$$

which concludes the proof. □

Remark 2.2.8. We note also that A^{op} is a skew PBW extension over R, where the elements of the Definition 1.1.4 are written in reverse order, and $\preceq^{\text{op}}$ is the order given by $\alpha \preceq^{\text{op}} \beta$ if and only if $\alpha^{\text{op}} \le \beta^{\text{op}}$. So, we can be that the set $\text{Mon}(A^{\text{op}}) = \{x_n^{\alpha_n} \cdots x_1^{\alpha_1} \mid \alpha^{\text{op}} = (\alpha_n, \dots, \alpha_1) \in \mathbb{N}^n\}$ is a free R-basis of A^{op}.

Proposition 2.2.9. *Let A be a bijective skew PBW extension of a Σ-rigid and ACCPL ring R. If R satisfies the ascending chain condition on left annihilators, then A is an ACCPL ring.*

Proof. From Proposition 2.2.7 we know that σ_i^{-1} is an automorphism of R^{op} and $\delta_i(\sigma_i^{-1}(r))$ is a σ_i^{-1}-derivation of R^{op}, for $i = 1, \dots, n$, and hence $A^{\text{op}} \simeq \sigma(R^{\text{op}})\langle x_1, \dots, x_n\rangle$ considering these families of automorphisms and derivations over R^{op}. In this way the assertion is due to Theorem 2.2.5. □

Corollary 2.2.10 ([31], Corollary 3.6)**.** *Let R be an ACCPL ring, σ a rigid automorphism of R and δ an σ-derivation of R. If R satisfies the ACC on left annihilators, then $R[x;\sigma,\delta]$ is an ACCPL ring.*

Future work

Leroy, Matczuk and Puczyłowski [22] gave a characterization of a ring $\mathbb{Z}$-graded R which is quasi-duo from R_0 and the quotient ring $R/A(R)$. As we saw, Suárez [45] defined the graded skew PBW extensions, so it is natural to think about how to apply this characterization to the context of these extensions. Of course, the difficulty of this purpose lies in the calculation of the mentioned quotient, so as a future work we will intend to calculate it with the aim of giving a complete classification of the skew PBW extensions that are and are not quasi-duo.

Bibliography

[1] Anderson, D. D., Anderson, D. F. and Zafrullah, M.,"Factorization in integral domains", *J. Pure Appl. Algebra.*, 69 (1990), 1-19.

[2] Bien M.H. and Öinert, J.,"Quasi-duo differential polynomial rings", *J. Algebra Appl.*, 17 (2018), No. 4, 1850072 (11 pages).

[3] Birkenmeier, F. and Tucci, P., "Homomorphic images and the singular ideal of a strongly right bounded ring", *Comm. Algebra*, 16 (1988), No. 6, 1099-1112.

[4] Courter, R. C., "Finite dimensional right duo algebras are duo", *Proc. Amer. Math. Soc.*, 84 (1982), 157-161.

[5] Curado, E. M. F., Hassouni, Y., Rego-Monteiro, M. A. and Rodrigues, L. M. C. S., "Generalized Heisenberg algebra and algebraic method: The example of an infinite square-well potential", *Phys. Lett. A.*, 372 (19) (2008), 3350-3355 .

[6] Dumitrescu, T., Al-Salihi, S. O. I., Radu, N. and Shah, T., "Some factorization properties of composite domains $A + XB[X]$ and $A + XB[[X]]$", *Comm. Algebra*, 28 (3) (2000), 1125-1139.

[7] Frohn, D., "A counterexample concerning ACCP in power series rings", *Comm. Algebra*, 30 (2002), 2961-2966.

[8] Frohn, D., "Modules with n-acc and the acc on certain types of annihilators", *J. Algebra*, 256 (2002), 467-483.

[9] Gallego, C. and Lezama, O., "Gröbner bases for ideals of σ-PBW extensions", *Comm. Algebra*, 39 (2011), No. 1, 50-75.

[10] Goodearl, K. R., "Prime ideals in skew polynomial rings and quantized Weyl algebras", *J. Algebra* 150 (1992), No. 2, 324-377.

[11] Grams, A., "Atomic domains and the ascending chain condition for principal ideals", *Math. Proc. Cambridge Philos. Soc.*, 75 (1974), 321-329.

[12] Heinzer, W. and Lantz, D.,"ACCP in polynomial rings: a counterexample", *Proc. Amer. Math. Soc.*, 121 (1994), 975-977.

[13] Hirano Y. , Hong C. H., Kim J. Y. and Park J. K., "On strongly bounded rings and duo rings", *Comm. Algebra* 23 (1995), No. 6, 2199-2214.

[14] Hong, C. Y., Kim, N. K. and Kwak, T. K., "Ore extensions of Baer and p.p.-rings", *J. Pure Appl. Algebra*, 151 (2000), 215-226 .

[15] Jacobson, N., "Finite-Dimensional Division Algebras over Fields", *Springer-Verlag*, (1996).

[16] Jonah, D., "Rings with the minimum condition for principal right ideals have the maximum condition for principal left ideals,", *Math. Z.*, 113 (1970), 106-112 .

[17] Krempa, J.,"Some examples of reduced rings", *Algebra Colloq.*, 3 (4) (1996), 289-300.

[18] Lam, T. Y. and Leroy, A.,"Algebraic conjugacy classes and skew polynomial rings", in: *Perspectives in Ring Theory* (Antwerp, 1987), in: *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, vol. 233, Kluwer, Dordrecht, 1988, 153-203.

[19] Lam, T. Y. and Leroy, A.,"Vandermonde and Wronskian matrices over division rings", *J. Algebra* 119 (1988), No. 2, 308-336.

[20] Lam, T. Y. and Leroy, A., "Principal one-sided ideals in Ore polynomial rings", in: *Algebra and Its Applications* (Athens, OH, 1999), in: *Contemp. Math.*, vol. 259, Amer. Math. Soc., Providence, RI, 2000, 333-352.

[21] Leroy, A., Matczuk, J., Puczyłowski, E. R., "Quasi-duo skew polynomial rings", *J. Pure and Appl. Alg.* 212 (2008), 1951-1959.

[22] Leroy ,A., Matczuk, J. and Puczyłowski, E. R., "A description of quasi-duo $\mathbb{Z}$-graded rings", *Comm. Algebra*, 38 (2010), 1319-1324.

[23] Lezama, O., Acosta, J. P. and Reyes, A., "Prime ideals of skew PBW extensions", *Rev. Un. Mat. Argentina* 56 (2015), No. 2, 39-55.

[24] Lezama, O. and Reyes, A., "Some Homological Properties of Skew PBW Extensions", *Comm. Algebra* 42 (2014), No. 3, 1200-1230.

[25] Marks, G., "A taxonomy of 2-primal rings", *J. Algebra* 266 (2003), No. 2, 494-520.

[26] Marks, G., "Duo rings and Ore extensions", *J. Algebra* 280 (2004), 463-471.

[27] Matczuk, J.,"A characterization of σ-rigid rings", *Comm. Algebra*, 32 (11) (2004), 4333-4336.

[28] Matczuk, J., "Ore extensions over duo rings", *J. Algebra* 297 (2006), 139-154.

[29] Mazurek, R. and Ziembowski, M.,"The ascending chain condition for principal left or right ideals of skew generalized power series rings", *J. Algebra*, 322 (2009), 983-994.

[30] Moussavi, A. and Hashemi, E.,"On (α, δ)-skew Armendariz rings", *J. Korean Math. Soc.*, 42(2) (2005), 353-363.

[31] Nasr-Isfahani, A. R., "The ascending chain condition for principal left ideals of skew polynomial rings", *Taiwanese J. Math.* 18 (3) (2014), 931-941.

[32] Niño, D. and Reyes, A."Some ring theoretical properties for skew Poincaré-Birkhoff-Witt extensions", *Bol. Mat. (N. S.)*, to appear (2017).

[33] Ore, O., "Theory of non-commutative polynomials", *Annals of Math.* 34 (1933), 480-508.

[34] Renault, G., "Sur des conditions de chaines ascendantes dans des modules libres", *J. Algebra*, 47 (1977), 268-275.

[35] Reyes, A., "Ring and module theoretical properties of skew PBW extensions", Thesis (Ph.D.), Universidad Nacional de Colombia, Bogotá, Colombia, 2013, 142 p.

[36] Reyes, A., "Gelfand-Kirillov Dimension of Skew PBW Extensions", *Rev. Col. Mat.* 47 (2013), No. 1, 95-111.

[37] Reyes, A., "Uniform Dimension over Skew PBW Extensions", *Rev. Col. Mat.* 48 (2014), No. 1, 79-96.

[38] Reyes, A., "Skew PBW Extensions of Baer, quasi-Baer, p.p. and p.q.-rings", *Rev. Integr. Temas Mat.* 33 (2015), No. 2, 173-189.

[39] Reyes, A. and Suárez, H., "Armendariz property for skew PBW Extensions and their classical ring of quotients", *Rev. Integr. Temas Mat.* 34 (2016), No. 2, 147-168.

[40] Reyes, A. and Suárez, H., "PBW bases for some 3-dimensional skew polynomial algebras", *Far East J. Math. Sci. (FJMS)* 101 (2017), No. 6, 1207-1228.

[41] Reyes, A. and Suárez, H., "σ-PBW Extensions of Skew Armendariz Rings", *Adv. Appl. Clifford Algebr.*, 27 (2017), 3197-3224, doi: 10.1007/s00006-017-0800-4.

[42] Reyes, A. and Suárez, H.,"Enveloping algebra and skew Calabi-Yau algebras over skew Poincaré-Birkhoff-Witt extensions", *Far East J. Math. Sci. (FJMS)*, 102 (2017), No. 2, 373-397.

[43] Reyes, A. and Suárez, H., "A notion of compatibility for Armendariz and Baer properties over skew PBW extensions", *Rev. Un. Mat. Argentina*, 59 (2018), No. 1, 157-178.

[44] Reyes, A. and Suárez, Y., "On the ACCP in skew Poincaré-Birkhoff-Witt extensions", *Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry*, to appear.

[45] Suárez, H. "Koszulity for graded skew PBW extensions", *Comm. in Algebra*, 45 (2017), No. 10, 4569-4580.

[46] Suárez, H., Lezama, O. and Reyes, A., "Some relations between N-Koszul, Artin-Schelter regular and Calabi-Yau algebras with skew PBW extensions", *Ciencia en Desarrollo* 6 (2015), No. 2, 205-213.

[47] Suárez, H. and Reyes, A., "A generalized Koszul property for skew PBW extensions", *Far East J. Math. Sci. (FJMS)* 101 (2017), No. 2, 301-320.

[48] Suárez, H., and Reyes, A., "Koszulity for skew PBW extensions over fields", *JP J. Algebra Number Theory Appl.*, 39 (2017), No. 2, 181-203.