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Around Superstability in Metric Abstract Elementary Classes

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To Bogota, Helsinki, Djursholm and São Paulo, cities where most of the results of this thesis were obtained.

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Resumen

En esta tesis estudiamos la unicidad de modelos límite como una noción débil de superestabilidad en clases elementales abstractas métricas, mostrando algunas consecuencias en el estudio de nociones de estabilidad geométrica como dominación, una noción débil de ortogonalidad y paralelismo, estudiando en algunos ejemplos los resultados obtenidos. Adicionalmente, estudiamos un teorema de transferencia de estabilidad bajo hipótesis de carácter local (superestabilidad) y docilidad.

Palabras clave: teoría de modelos, clases métricas, clases no elementales, superestabilidad, estabilidad, dominación, ortogonalidad, paralelismo.

Abstract

In this thesis, we study uniqueness of limit models as a weak version of superstability in metric abstract elementary classes, showing some consequences in some basic notions of geometric stability theory as domination, a weak notion of orthogonality and parallelism, studying the got results in some examples. Also, we study a stability transfer theorem under local character (superstability) and tameness in the setting of metric abstract elementary classes.

Keywords: model theory, metric classes, non-elementary classes, superstability, stability, domination,orthogonality, parallelism.

Contents

Introduction

Metric Abstract Elementary Class (shortly, MAECs) is a notion which corresponds to a kind of amalgam of the notions of *Abstract Elementary Class* (see [She99, Bal09]) and elementary classes in the setting of *Continuous Logic*. This notion has been studied by S. Shelah and A. Usvyatsov (see [SU]) and A. Hirvonen and T. Hyttinen (see [HH09]). MAECs are important for many reasons. Mainly, this setting provides a suitable abstract framework for studying classes of complete metric structures which cannot be studied as CFO elementary classes. CFO requires that all considered operators satisfy some restricted properties like boundedness, compactness and uniform continuity, but there are lots of metric structures studied in Physics (for example, *rigged Hilbert spaces* in Quantum Mechanics) which do not satisfy those conditions.

In [HH09], Hirvonen and Hyttinen studied a categoricity transfer theorem in the setting of *Homogeneous Metric Abstract Elementary Classes*. In that paper, Hirvonen and Hyttinen introduced a notion of independence which under ω-d-stability is well-behaved. There is no suitable study of geometric stability in those particular examples. However, Hirvonen and Hyttinen defined *domination* in *homogeneous* MAEC, and used it to prove unidimensionality of Galois types in this setting, and then used it for the proof of their version of *categoricity transfer theorem*. Also, they did a suitable analysis of ω-d-stability in their setting, but they did not study more general consequences under the weaker superstability assumptions.

Uniqueness (up to isomorphism) of limit models is a *robust* version of superstability which R. Grossberg and M. VanDieren used for giving a partial answer -in the setting of *tame* AEC- to the following conjecture due to Shelah: For any *(discrete) abstract elementary class* (for short AEC) $\mathcal{K}$ there exists a cardinality μ (which depends on the Löwenheim-Skolem number of $\mathcal{K}$) such that if $\mathcal{K}$ is λ-categorical for some $\lambda \geq \mu$ then $\mathcal{K}$ is κ-categorical for all $\kappa \geq \mu$. R. Grossberg, M. VanDieren and A. Villaveces proved in [GVV08] *uniqueness (up to isomorphism) of limit models* assuming that $\mathcal{K}$ does not allow long splitting chains, $\mathcal{K}$ satisfies locality of splitting and $\mathcal{K}$ is κ-Galois-stable (which follow from categoricity).

In [Bal0x], J. Baldwin does a study of a weak notion of domination which is based on a rough notion of independence in terms of on intersections of models, although he assumes uniqueness of limit models as a superstability-like assumption.

In superstable first order theories, getting a decomposition (up to equi-domination) of stationary types as a finite product of regular types provides us a proof of the following fact due to Lachlan: A countable superstable theory has 1 or infinitely many countable models. Also, there are versions of this decomposition theorem in not necessarily stable theories (see [OU09b]) such as rosy and dependent theories (see [OU09a]), settings where there is a very well-behaved independence notion.

In the first chapter, we provide some basic definitions and facts about *Metric Abstract Elementary Classes*. In section 1.4 we introduce a notion of independence (*smooth independence*) which is well-behaved under stability assumptions. In this section, we prove some properties which were not considered in [HH09] such as a version of transitivity of independence. Also, we prove a new property on independence notions which we called *continuity*. In general, this result says that if $(b_n) \to b$ and every b_n is independent from some model M over N, so is b. This property is very useful in the proof of uniqueness of limit models in the setting of MAECs. In section 1.5 we introduce a notion of κ-d-tame MAEC and we introduce a variant of s-independence (called tame-independence) and prove some basic properties of this independence notion. We use this independence notion towards getting a *stability transfer theorem* in the setting of κ-d-tame MAEC (in a similar way as [BKV06]).

In chapter 2, we make some steps towards getting uniqueness of limit models under superstability-like assumptions. Although the sketch of the proof which we do in this work is quite similar to the proof given in [GVV08], we have to point out that various steps of the proof in our setting are quite different from the details given by Grossberg, VanDieren and Villaveces. We even have to give a different notion of *reduced tower* based in our notion of independence.

In chapter 3, we introduce a notion of dominance in the setting of superstable MAEC. We base our work on [Bal0x], but we define our notion of dominance using smooth independence and not just using intersections as J. Baldwin does in his paper. We prove that under suitable assumptions, given a tuple $(M, \mathcal{M}, \overline{a}, N)$ (where $\mathcal{M}$ is a resolution of M which witnesses that M is a limit model over some model M_0) such that $\overline{a} \downarrow_{M_0}^{\mathcal{M}_0} M$ (and therefore ga-tp$(\overline{a}/M)$ is a stationary type because M is an universal model over M_0), there exist N^* and a resolution $\mathcal{M}^*$ which witnesses that M is a limit model over M_0 such that $\overline{a} \bowtie_M^{\mathcal{M}^*} N^*$. Also, in this chapter we study notions of orthogonality and parallelism in superstable MAECs, inspired in [She09b]. In this study, we drop some strong conditions given in [She09b] and simplify some of the proofs given there. Also, we prove some properties which were not studied in [She09b].

In chapter 4 we prove a *stability transfer theorem* in the setting of κ-d-tame MAEC, using the independence notion defined in section 1.5 of chapter 1. Discrete *tame* AECs are a very special kind of AECs which have a categoricity transfer theorem (see [GV06a]) and a nice stability transfer theorem (see [BKV06]). In fact -under $\aleph_0$-tameness and $\aleph_0$-locality (assuming $LS(\mathcal{K}) = \aleph_0$)-, J. Baldwin, D. Kueker and M. VanDieren proved that $\aleph_0$-Galois-stability implies κ-Galois-stability for every cardinality κ. First, they prove that $\aleph_0$-Galois-stability implies $\aleph_n$-Galois stability for every $n < \omega$ (in fact, their argument works for getting κ-Galois-stability if $cf(\kappa) > \omega$) and so (by ω-locality) $\aleph_\omega$-Galois-stable (where that argument works for getting κ-Galois stability if $cf(\kappa) = \omega$). In this thesis, we prove an analogous stability transfer theorem in a metric version of tameness of MAECs, but we use a version of local character of a notion of independence (tamely independence) which is well-behaved in that setting instead of using ω-locality as in [BKV06].

In chapter 5, we study some examples and interpret the results on geometric stability theory studied in chapter 3. First, we consider Hilbert spaces with a unitary operator. Secondly, we study L^p spaces. Then, we study Hilbert spaces with an unbounded closed self-adjoint operator. Finally, we study Gelfand triplets as MAECs.

Index of Symbols

CHAPTER 1

Some independence notions in Metric Abstract Elementary Classes

1.1 Some basic definitions and results

Abstract Elementary Class is a notion due to B. Jónsson and S. Shelah (see [Jón56, Jón60, She87a, She87b]) which corresponds to a generalization of *first order elementary classes* and was used initially for studying the model-theoretical behavior (categoricity spectrum, stability spectrum, etc.) of non (first order) elementary classes like the class of models of an $L_{\omega_1,\omega}$ sentence. But this approach is not sufficient for studying analytic structures such as those arising in *Operator Theory*, *Functional Analysis*, *Probability*, etc.

Continuous logic (for short, *CL*) is a logic which is used for studying these kinds of analytical structures (see [BYBHU08]) due originally to W. Henson and J. Iovino (*Positive Bounded Theories*, see [HI02]; inspired in works of C. C. Chang and H. J. Keisler, see [KC66]) and in a later version due to I. Ben-Yaacov and A. Usvyatsov. In this logic, we consider structures with uniformly continuous functions and relations and a new particular symbol which is interpreted as a metric in every sort in the structures considered. Metrics play the same role in this setting as equality plays in first order extensions. In CL, formulas are defined inductively, taking values continuously between 0 and 1 (where their values are interpreted as the "distance to the truth"). We do not provide either the inductive definition of formulas or the definition of "satisfaction" in this work, but we refer to [BYBHU08] for basic definitions and facts in this logic.

Although CL provides a nice framework for studying analytical structures, there is the obvious limitation that most known elementary examples in CL impose very restrictive condition on operators (as boundness and compactness), because these requirements are necessary for constructing ultraproducts towards getting an axiomatization in CL. This excludes lots of interesting examples which cannot be studied in CL, e.g. Hilbert spaces with an unbounded closed self-adjoint operator (see [Arg1X]). So, it is necessary to set a more general framework to be able to study this kind of examples.

Metric Abstract Elementary Class is a framework due to S. Shelah and A. Usvyatsov (see [SU]) and independently by Å. Hirvonen and T. Hyttinen (see [HH09]), which corresponds to a kind of amalgam of Abstract Elementary Classes (for short, AECs) and elementary classes in the context of CL.

Definition 1.1.1. The *density character* of a topological space is the smallest cardinality of a dense subset of the space. If X is a topological space, we denote its density character by $dc(X)$. If A is a subset of a topological space X, we define $dc(A) := dc(\overline{A})$, where $\overline{A}$ denotes the topological closure of A relative to X. Notice that this notion depends on X, but there is no ambiguity if we are working in a fixed context.

Fact 1.1.2. *Let $\kappa > \omega$ be a regular cardinal and M be a metric space of density character κ. Let $\{a_i : i < \kappa\}$ be a dense subset of M. Then there exist $\varepsilon > 0$ and $Y \subset_\kappa \kappa$ (i.e.: $|Y| = \kappa$) such that $d(a_i, a_j) \geq \varepsilon$ for every $i \neq j \in Y$.*

Proof. If this fact were false, given $r \in \mathbb{Q}^+$ there are at most $\lambda_r < \kappa$ many indexes $i, j < \kappa$ such that $d(a_i, a_j) \geq r$. Denote by X_r a maximal set of such indexes of size $< \kappa$ (such maximal set exists by Zorn's lemma, because $cf(\kappa) = \kappa$ and there are at most $\lambda_r < \kappa$ many indexes $i, j < \kappa$ such that $d(a_i, a_j) \geq r$). Define $X := \bigcup_{r\in\mathbb{Q}^+} X_r$. Since $cf(\kappa) > \omega$, notice that $|X| < \kappa$.

We claim that $\{a_i : i \in X\}$ is dense in M. Let $a \in M$ and $r \in \mathbb{Q}^+$. Since $\{a_i : i < \kappa\}$ is a dense subset of M, there exists $i < \kappa$ such that $d(a, a_i) < r/2$. If $i \in X$, we are done. If not, since $i \notin X$, in particular $i \notin X_{r/2}$. So, there exists $i_r \in X_{r/2}$ such that $d(a_i, a_{i_r}) < r/2$ (since we chose every X_r maximal). By the triangle inequality, $d(a, a_{i_r}) \leq d(a, a_i) + d(a_i, a_{i_r}) < r/2 + r/2 = r$, so we are done. Therefore, X is a dense subset of M of size $< \kappa$ (contradiction).

$\square_{\text{Fact 1.1.2}}$

We consider a natural adaptation of the notion of *Abstract Elementary Class* (see [Gro02] and [Bal09]), but work in a context of Continuous Logic that generalizes the "First Order Continuous" setting of [BYBHU08] by removing the assumption of uniform continuity[1] and we consider closed operators instead of uniform continuity. Our definitions are inspired in the work done by Åsa Hirvonen and Tapani Hyttinen (see [HH09]).
Next, we define the notion of structure that we use in this thesis. This is an adaptation of the notion of structure given in [BYBHU08], but we do not require to consider uniform continuity in this thesis, as in [BYBHU08], because we are not interested to get axiomatizations of the classes. However, we require that relational and functional symbols satisfy some closedness assumptions.

[1]Uniform continuity guarantees logical compactness in their formalization, but we drop compactness in our more general setting of AEC.

Definition 1.1.3. Let L be a language as in [BYBHU08], but without the uniform continuity modulus. A multi-sorted *metric* L-*structure* is a tuple

$$M := (\{\langle A_i, d_i\rangle\}_{i\in I}, \mathbb{R}, \{c_j\}_{j\in J}, \{F_k\}_{k\in K}, \{R_l\}_{l\in L}), \text{ where:}$$

1. Each (A_i, d_i) is a complete metric space.
2. $\mathbb{R}$ is an isomorphic copy of the real field $(\mathbb{R}, +, \cdot, 0, 1, \le)$.
3. Each c_j is a constant in a fixed sort $A_{i(j)}$.
4. Each R_k is a continuous predicate, which corresponds to a function $R_k : A_{i(k)_1} \times \cdots \times A_{i(k)_n} \to [0, 1]$ which is weakly closed, i.e.: if $(\overline{x})_{n<\omega} \to \overline{x}$ as tuples, then $(R_k(\overline{x}_n))_{n<\omega} \to R_k(\overline{x})$, where n is called the *arity of* R_k.
5. Each F_l is a function $F_l : A_{i(l)_1} \times \cdots \times A_{i(l)_m} \to A_{i(l)}$ which is weakly closed(i.e.: if $(\overline{x})_{n<\omega} \to \overline{x}$ as tuples, then $(F_l(\overline{x}_n))_{n<\omega} \to F_l(\overline{x})$), where m is called the *arity of* F_l .

If it is clear that we are working in a metric context, we just called them L-*structures*.

Definition 1.1.4. Let $\mathcal{K}$ be a class of (possibly multi-sorted) metric L-structures and $\prec_{\mathcal{K}}$ be a binary relation defined in $\mathcal{K}$. We say that $(\mathcal{K}, \prec_{\mathcal{K}})$ is a *Metric Abstract Elementary Class* (shortly *MAEC*) if:

1. $\mathcal{K}$ and $\prec_{\mathcal{K}}$ are closed under isomorphism.
2. $\prec_{\mathcal{K}}$ is a partial order in $\mathcal{K}$.
3. If $M \prec_{\mathcal{K}} N$ then M is an L-substructure of N -denoted by $M \subseteq N$-.
4. (*Tarki-Vaught chains*) If $(M_i : i < \lambda)$ is a $\prec_{\mathcal{K}}$-increasing chain then
   a) the function symbols in L can be uniquely interpreted on the completion of $\bigcup_{i<\lambda} M_i$ in such a way that $\overline{\bigcup_{i<\lambda} M_i} \in \mathcal{K}$
   b) for each $j < \lambda$, $M_j \prec_{\mathcal{K}} \overline{\bigcup_{i<\lambda} M_i}$
   c) if each $M_i \prec_{\mathcal{K}} N$, then $\overline{\bigcup_{i<\lambda} M_i} \prec_{\mathcal{K}} N$.
5. (*Coherence*) if $M_1 \subseteq M_2 \prec_{\mathcal{K}} M_3$ and $M_1 \prec_{\mathcal{K}} M_3$, then $M_1 \prec_{\mathcal{K}} M_2$.
6. (Downward Löwenheim-Skolem axiom) There exists a cardinality $LS(K)$ (which is called the *metric Löwenheim-Skolem number* of $\mathcal{K}$) such that if $M \in \mathcal{K}$ and $A \subseteq M$, then there exists $N \in \mathcal{K}$ such that $dc(N) \le dc(A) + LS(K)$ and $A \subseteq N \prec_{\mathcal{K}} M$.

Examples 1.1.5. 1. Any continuous elementary class (see [BYBHU08]) with the usual elementary substructure relation is an MAEC. Important cases include

a) Hilbert spaces with a unitary operator (Argoty and Berenstein, see [AB09]).
b) Nakano spaces with compact essential rank (Poitevin, see [Poi06]).
c) Probability Spaces with an automorphism, see [BH04].
d) Compact Abstract Theories, see [BY03, BY05]

2. Nakano spaces with non-compact essential rank, see [HH11].

3. A subclass of completions of metric spaces which approximately satisfy a positive bounded theory, where $\prec_{\mathcal{K}}$ is interpreted by the approximate elementary submodel relation (see [HI02]).

4. Gelfand triplets (see section 5.4).

5. Any (discrete) AEC is an MAEC together with the discrete metric.

Definition 1.1.6. A $\prec_{\mathcal{K}}$-increasing chain $\langle M_i : i < \alpha\rangle$ is said to be *continuous* if and only if for every limit ordinal $\beta < \alpha$ we have that $M_\beta = \overline{\bigcup_{\gamma<\beta} M_\gamma}$.

Definition 1.1.7 (Resolution). We say that a $\prec_{\mathcal{K}}$-increasing and continuous chain $\langle M_i : i < \alpha\rangle$ is a *resolution* of M if and only if $\overline{\bigcup_{i<\alpha} M_i} = M$.

Definition 1.1.8. We call a function $f : M \to N$ a $\mathcal{K}$-*embedding* if

1. For every k-ary function symbol F of L, we have

$$f(F^M(a_1, \cdots, a_k)) = F^N(f(a_1), \cdots, f(a_k))$$

In particular, f is an isometry.

2. For every constant symbol c of L, $f(c^M) = c^N$.

3. For every m-ary relation symbol R of L, for every $\bar{a} \in M^m$, $d(\bar{a}, R^M) = d(f(\bar{a}), R^N)$.

4. $f[M] \prec_{\mathcal{K}} N$ (notice that $f[M] \in \mathcal{K}$ by axiom 1. of definition 1.1.4).

1.2 The Shelah Presentation Theorem in MAECs.

As in (discrete) AECs, we have a version of the *Shelah Presentation Theorem*. In [Hir06], Å. Hirvonen proved a version of this theorem in order to provide the construction of Ehrenfeucht-Mostowski models in MAECs, as in (discrete) AECs. However, she did not prove that MAECs are PC-classes. Here, we prove that MAECs are in fact PC-classes. In fact, she proved the following result:

Theorem 1.2.1 (Hirvonen). *Let $(\mathcal{K}, \prec_{\mathcal{K}})$ an MAEC of L-structures with $|L| + LS(\mathcal{K}) = \aleph_0$. Then for each $M \in \mathcal{K}$ we can define an expansion M^* with Skolem functions F^k_n $(k, n < \omega)$ such that:*

1. *If $A \subset M^*$ and A is closed under the functions F^k_n then $\overline{A} \upharpoonright L \in \mathcal{K}$ and $\overline{A} \upharpoonright L \prec_{\mathcal{K}} M$.*
2. *For all finite tuples $\overline{a} \in M$, $A_{\overline{a}} := \{(F_n^{\text{length}(\overline{a})})^{M^*}(\overline{a}) : n < \omega\}$ is such that*
   a) *$\overline{A_{\overline{a}}} \upharpoonright L \in \mathcal{K}$ and $\overline{A_{\overline{a}}} \upharpoonright L \prec_{\mathcal{K}} M$,*
   b) *If $\overline{b} \subseteq \overline{a}$ (as sets) then $\overline{a} \in A_{\overline{b}} \subset A_{\overline{a}}$.*

Although our proof in the metric setting follows the same sketch of the proof given in the discrete case, we have to point out that the details are quite different because we are working in a metric setting. We have to use the formulation of Continuous Logic, clarifying that we drop the uniform continuity of the function and relational symbols.

We mention some basic (and classic) facts towards getting a proof of Shelah Presentation Theorem. These basic facts are also used in the classic proof in the (discrete) Abstract Elementary Classes (for short, AECs), but for the sake of completeness we provide their statements.

Fact 1.2.2. *Let $(I, \leq)$ be a directed partial order of size λ. Then there exists a family $\{I_\alpha : \alpha < \lambda\}$ of suborders of I such that:*

1. *Each I_α is a directed order and $|I_\alpha| < \lambda$*
2. *If $\alpha < \beta < \lambda$, then $I_\alpha \leq I_\beta$*
3. *$I = \bigcup_{\alpha<\lambda} I_\alpha$.*

We prove the following fact in a similar way as in (discrete) AECs (mutatis mutandis). In fact, we strongly use the Tarski-Vaught chains axiom (MAEC axiom). Notice that in MAECs, this axiom involves not just the union of the $\prec_{\mathcal{K}}$-chain, we have to take the completion of that union. In spite of the sketch of the proof being almost the same as in (dicrete) AECs, for the sake of completeness we provide a proof of this fact.

Proposition 1.2.3. *Let $(I, \leq)$ be a directed partial order and $(M_i : i \in I)$ a $\prec_{\mathcal{K}}$-directed system. Then:*

(a) $\overline{\bigcup_{i\in I} M_i} \in \mathcal{K}$.

(b) $M_j \prec_{\mathcal{K}} \overline{\bigcup_{i\in I} M_i}$ *for each* $j \in I$.

(c) *If* $N \in \mathcal{K}$ *and* $M_j \prec_{\mathcal{K}} N$ *for each* $j \in I$, *then* $\overline{\bigcup_{i\in I} M_i} \prec_{\mathcal{K}} N$.

Proof. Assume this fact holds for $\alpha < |I|$. By fact 1.2.2 we have that there exists a family $\{I_\alpha : \alpha < \lambda\}$ of suborders of I such that:

1. Each I_α is a directed order and $|I_\alpha| < |I|$
2. If $\alpha < \beta < |I|$, then $I_\alpha \leq I_\beta$
3. $I = \bigcup_{\alpha<|I|} I_\alpha$.

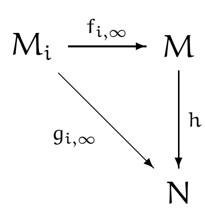
Define $M_\alpha := \overline{\bigcup_{i\in I_\alpha} M_i}$. By induction hypothesis (b) we have that $M_j \prec_{\mathcal{K}} M_\alpha$ for every $j \in I_\alpha$. If $\alpha < \beta$, since $I_\alpha \subseteq I_\beta$ then $M_j \prec_{\mathcal{K}} M_\beta$ for every $j \in I_\alpha$. By induction hypothesis (c) we have that $M_\alpha := \overline{\bigcup_{j\in I_\alpha} M_j} \prec_{\mathcal{K}} M_\beta$.

It is easy to check that $\overline{\bigcup_{\alpha<|I|} M_\alpha} = \overline{\bigcup_{i\in I} M_i}$, so by definition 1.1.4 4) (a) we have that $\overline{\bigcup_{i\in I} M_i} = \overline{\bigcup_{\alpha<|I|} M_\alpha} \in \mathcal{K}$. Then (a) holds.
If $j \in I$, there exists $\alpha < |I|$ such that $j \in I_\alpha$, so $M_j \prec_{\mathcal{K}} M_\alpha$ (by induction hypothesis) and by definition 1.1.4 (4) (b) $M_\alpha \prec_{\mathcal{K}} \overline{\bigcup_{\alpha<|I|} M_\alpha} = \overline{\bigcup_{i\in I} M_i}$. Therefore $M_j \prec_{\mathcal{K}} \overline{\bigcup_{i\in I} M_i}$, i.e. (b) holds.
Let N be an L-structure in $\mathcal{K}$ such that $M_j \prec_{\mathcal{K}} N$ for each $j \in I$. By induction hypothesis (c), for each $\alpha < |I|$ we have that $M_\alpha \prec_{\mathcal{K}} N$. So, by definition 1.1.4 (4) (c) we have that $\overline{\bigcup_{i\in I} M_i} = \overline{\bigcup_{\alpha<|I|} M_\alpha} \prec_{\mathcal{K}} N$. So, (c) holds. □

Definition 1.2.4 (directed system). Let $\mathcal{K}$ be a Category. A functor $D : (I, \leq) \to \mathcal{C}$ is said to be a *directed system* if and only if $(I, \leq)$ is a directed ordered set. Set $M_k := D(k)$ for every $k \in I$ and $f_{i,j} := D((i, j)) : M_i \to M_j$ the morphism associate to the unique I-morphism $(i, j) : i \to j$ via D whenever $i \leq_I j$.

Definition 1.2.5 (directed limits). We say that $\mathcal{K}$ is *closed under directed limits* iff for every directed system $D : (I, \leq) \to \mathcal{C}$ there exist $M \in ob(\mathcal{K})$ and $\mathcal{C}$-morphisms $f_{i,\infty} : M_i \to M$ $(i \in I)$ such that

1. for any $i \leq_I j$ we have $f_{i,\infty} = f_{j,\infty} \circ f_{i,j}$
2. if any $N \in ob(\mathcal{C})$ has a system of $\mathcal{C}$-morphisms $g_{i,\infty} : M_i \to N$ which safisties 1. above, then there exists a unique $\mathcal{C}$-morphism $h : M \to N$ such that $g_{i,\infty} = h \circ f_{i,\infty}$.



Such morphisms $f_{i,\infty}$ are called *canonical morphisms.*

Corollary 1.2.6. *An MAEC $\mathcal{K}$ (viewed as a category with morphisms the $\mathcal{K}$-embeddings) is closed under directed limits.*

Proof. This is a direct consequence of proposition 1.2.3 and definition 1.1.4 (1). □Corollary 1.2.6

And now, we provide the proof of Shelah Presentation Theorem in MAECs. Although the sketch of this proof is quite similar to the proof of the original Shelah Presentation Theorem in (discrete) AECs, we have to point out that we strongly use continuity and density, because our setting is metric complete structures. Also, the theory and syntactic types used in the proof are not in *Classical First Order Logic* but in *CL* (although without uniform continuity assumptions).

Theorem 1.2.7 (Shelah Presentation Theorem in MAECs). *Given an MAEC* $(\mathcal{K}, \prec_{\mathcal{K}})$*, there exist an expansion* L' *of* $L(\mathcal{K})$*, a* L'*-theory* T' *(as in CL but considering weakly closed formulas instead of uniformly continuous formulas) and a set of* T'*-types* Γ *such that*

$$\mathcal{K} = PC_L(T', \Gamma) \;\; := \;\; \{M : \; M \textit{ is an } L\textit{-structure which has an } L'\textit{- expansion } M' \textit{ such that } M' \models T' \textit{ and } M' \textit{ omits all } p \in \Gamma\}$$

i.e.: $\mathcal{K}$ *is a continuous projective class with omitting types (for short, a continuous* $PC\Gamma$ *class).*

Proof. Let L' be the language obtained from $L(\mathcal{K})$ by adding new n-ary function symbols F_i^n $(i < LS(\mathcal{K}))$. Let T' be the theory which says that

$$\sup x_0 \dots \sup x_{n-1} |F_i^n(x_o, \cdots, x_{n-1}) - x_i| = 0$$

for all $i < n$. Notice that all F_i^n are defined as projections, so they are continuous (and therefore, weakly closed).
Take $M' \models T'$ and $\overline{a} \in M'$. For $\overline{b} <_{\text{subtuple}} \overline{a}$, define $U_{\overline{b}} := \{F_i^m(\overline{b}) : i < LS(\mathcal{K})\}$, where $m := l(\overline{b})$.

Define Γ as follows: for every $M' \models T'$ and a fixed submodel $N \subset M'$ of density character $LS(\mathcal{K})$ and for every tuple $\overline{a} \in M'$, $tp_{L'}(\overline{a}/\emptyset) \in \Gamma$ (the L'-syntactic type) **unless** we have the following two conditions:

1. For every $\overline{b} \leq_{\text{subtuple}} \overline{a}$, $U_{\overline{b}}$ is a dense subset of a submodel of $M' \restriction L$, which we denote by $M_{\overline{b}}$, and $M_{\overline{b}} \in \mathcal{K}$.

2. If $\overline{b} \leq_{\text{subtuple}} \overline{a}$, we have $M_{\overline{b}} \prec_{\mathcal{K}} M_{\overline{a}}$.

Let $M \in PC_L(T', \Gamma)$. So, there exists $M' \models T'$ such that omits all the types in Γ and $M = M' \restriction L$. Consider the sets $U_{\overline{a}}$, for each $\overline{a} \in M$. Since M' omits all the types in Γ, each $U_{\overline{a}}$ is a dense subset of the universe of a submodel $M_{\overline{a}}$ of M such that $M_{\overline{a}} \in \mathcal{K}$. By proposition 1.2.3 we have that $\overline{\bigcup_{\overline{a} \in M} M_{\overline{a}}} \in \mathcal{K}$.
Since $M' \models T'$, we have that $\overline{a} \in U_{\overline{a}}$ and so $\overline{\bigcup_{\overline{a} \in M} M_{\overline{a}}} = M$. Therefore $M \in \mathcal{K}$.

In the other direction, take $M \in \mathcal{K}$. We define M' as follows: for $n = 0$, choose $M_\emptyset \prec_{\mathcal{K}} M$ of density character $LS(\mathcal{K})$ and let $U_\emptyset := \{F_i^0 : i < LS(\mathcal{K})\}$ be an enumeration of a dense subset of

$M_\emptyset$. Having done this for n, let $\overline{a} \in M$ be of length $n+1$. Choose $M_{\overline{a}} \prec_{\mathcal{K}} M$ of density character $LS(\mathcal{K})$ which contains $\bigcup\{M_{\overline{b}} : \overline{b} <_{subtuple} \overline{a}\}$ and $\overline{a}$, and let $U_{\overline{a}} := \{F_i^{n+1}(\overline{a}) : i < LS(\mathcal{K})\}$ be an enumeration of a dense subset of $M_{\overline{a}}$ such that $F_i^{n+1}(\overline{a}) = a_i$ for $i < n+1$.
So, $M' \models T'$ and M' omits every type in Γ. Hence, $M \in PC_L(T', \Gamma)$. $\square_{\text{Th. 1.2.7}}$

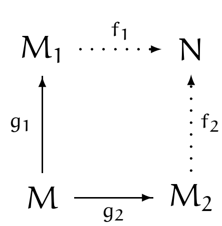
Notation 1.2.8. A (discrete) class $\mathcal{K}$ is said to be $PC\Gamma(\lambda, \mu)$ iff $\mathcal{K}$ is of the form $PC_L(T, \Gamma)$ for $|T| \leq \lambda$ and $|\Gamma| \leq \mu$.

Question 1.2.9. Let L be a (first order) language and $\mu := \sup\{|L|, \kappa\}$. Since Shelah proved that any AEC is a $PC\Gamma(\kappa, 2^\kappa)$ class with $LS(\mathcal{K}) = \kappa$, as a consequence of the M. Morley's omitting types theorem -see [Mor65]- we have that there exists a cardinality $H_1 := \beth_{(2^\mu)^+}$ such that if $\mathcal{K}$ is an AEC of L-structures with $LS(\mathcal{K}) = \kappa$ such that if there exists $M \in \mathcal{K}$ of cardinality $> H_1$ then there exists a model in $\mathcal{K}$ in any cardinality $> H_1$. But its existence strongly depends on the existence of Hanf numbers in (discrete) omiting type classes (which depends on infinitary logics, see [Mor65]). In the metric case, we do not have a suitable study of metric infinitary logics. It is still open if in a suitable notion of metric infinitary logic we can do a suitable analysis of Hanf numbers which implies that metric PC classes have a Hanf number, and so MAECs do as well.

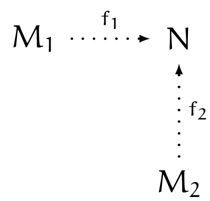
1.3 More basic definitions and results

Assumption 1.3.1. *Throughout this thesis, we assume that $\mathcal{K}$ is an MAEC such that $\mathcal{K}$ has a Hanf number (i.e., there exists a cardinal $Hanf(\mathcal{K})$ such that if there exists a model $M \in \mathcal{K}$ with density character $> Hanf(\mathcal{K})$, then there exist arbitrarily large enough models in $\mathcal{K}$).*

Definition 1.3.2 (Amalgamation Property, AP). Let $\mathcal{K}$ be an MAEC. We say that $\mathcal{K}$ satisfies the *Amalgamation Property* (for short *AP*) if and only if for every $M, M_1, M_2 \in \mathcal{K}$, if $g_i : M \to M_i$ is a $\mathcal{K}$-embedding $(i \in \{1,2\})$ then there exist $N \in \mathcal{N}$ and $\mathcal{K}$-embeddings $f_i : M_i \to N$ $(i \in \{1,2\})$ such that $f_1 \circ g_1 = f_2 \circ g_2$.



Definition 1.3.3 (Joint Embedding Property, JEP). Let $\mathcal{K}$ be an MAEC. We say that $\mathcal{K}$ satisfies the *Joint Embedding Property* (for short *JEP*) if and only if for every $M_1, M_2 \in \mathcal{K}$ there exist $N \in \mathcal{N}$ and $\mathcal{K}$-embeddings $f_i : M_i \to N$ $(i \in \{1,2\})$.



Remark 1.3.4. Notice that if $\mathcal{K}$ has a prime model (i.e., a model $N \in \mathcal{K}$ such that for every $M \in \mathcal{K}$ there exists a $\mathcal{K}$-embedding $f : N \to M$), then AP implies JEP.

Remark 1.3.5 (Monster Model). If $\mathcal{K}$ is an MAEC which satisfies AP and JEP and has a model of density character $> \mathrm{Hanf}(\mathcal{K})$ (i.e., $\mathcal{K}$ has large enough models), then we can construct a large enough model $\mathbb{M}$ (which we call a *Monster Model*) which is homogeneous–i.e., every isomorphism between two $\mathcal{K}$-substructures of $\mathbb{M}$ can be extended to an automorphism of $\mathbb{M}$– and also universal –i.e., every model with density character $< dc(\mathbb{M})$ can be $\mathcal{K}$-embedded into $\mathbb{M}$.

Definition 1.3.6 (Galois type). Under the existence of a monster model $\mathbb{M}$ as in Remark 1.3.5, for all $\overline{a} \in \mathbb{M}$ and $N \prec_{\mathcal{K}} \mathbb{M}$, we define $\text{ga-tp}(\overline{a}/N)$ (the *Galois type of* $\overline{a}$ *over* N) as the orbit of $\overline{a}$ under $Aut(\mathbb{M}/N) := \{f \in Aut(\mathbb{M}) : f \upharpoonright N = id_N\}$. We denote the set of Galois types over a model $M \in \mathcal{K}$ by $\text{ga-S}(M)$.

Although it is easier to see Galois types as orbits under automorphisms of $\mathbb{M}$, for the sake of completeness we provide the proof of the following equivalence which we use to prove later the existence of universal models. In fact, the following definition of Galois type was the original one considered by Shelah. The two definitions are equivalent under AP, JEP and the existence of models of size $\geq \mathrm{Hanf}(\mathcal{K})$.

Proposition 1.3.7. *Assume there exists a monster model* $\mathbb{M}$ *as in remark 1.3.5. Let* $M \in \mathcal{K}$ *and* $a, b \in \mathbb{M}$. $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$ *iff there exists* $M_1, M_2, N \in \mathcal{K}$ *and* $\mathcal{K}$*-embeddings* $f_1 : M_1 \to N$ *and* $f_2 : M_2 \to N$ *such that* $a \in M_1$, $b \in M_2$, $f_1(a) = f_2(b)$ *and* $f_1 \upharpoonright M = f_2 \upharpoonright M = id_M$ *(which we denote by* $(M_1, a, M) \sim (M_2, b, M)$*).*

Proof. $(\Rightarrow)$ If there exists $f \in Aut(\mathbb{M}/M)$ such that $f(a) = b$, let $M_1 \succ_{\mathcal{K}} M$ be such that $a \in M_1$ and $M_2 \supset f[M_1] \cup \{b\}$ be a model in $\mathcal{K}$. Considering $N := M_2$, $f_1 := f \upharpoonright M_1 : M_1 \to N$ and $f_2 := id_{M_2}$ we are done.

$(\Leftarrow)$ Suppose there exists $M_1, M_2, N \in \mathcal{K}$ and $\mathcal{K}$-embeddings $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ such that $a \in M_1$, $b \in M_2$, $f_1(a) = f_2(b)$ and $f_1 \upharpoonright M = f_2 \upharpoonright M = id_M$. By the homogeneity of $\mathbb{M}$, let f be an automorphism of $\mathbb{M}$ which extends $f_2^{-1} \circ f_1$. Notice that $f(a) = f_2^{-1} \circ f_1(a) = f_2^{-1} \circ f_2(b) = b$ and $f \upharpoonright M = id_M$, so $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$. $\square_{\text{Prop. 1.3.7}}$

Remark 1.3.8. Under AP, $\sim$ defined in proposition 1.3.7 is an equivalence relation, and $(M_1, a, M)/\sim$ is called the *Galois type* of a over M inside M_1 (which we denote by $\text{ga-tp}(a/M, M_1)$).

Throughout this paper, we assume the existence of a model-homogeneous and universal monster model as in Remark 1.3.5.

Definition 1.3.9 (Distance between types). Let $p, q \in$ ga-S(M). We define $d(p, q) := \inf\{d(\overline{a}, \overline{b}) : \overline{a}, \overline{b} \in \mathbb{M}, \overline{a} \models p, \overline{b} \models q\}$, where $lg(\overline{a}) = lg(\overline{b}) =: n$ and $d(\overline{a}, \overline{b}) := \max\{d(a_i, b_i) : 1 \leq i \leq n\}$.

The following facts are easy computations that almost directly follow from the definitions. We provide their proofs for the sake of completeness.

Fact 1.3.10 (Hyttinen-Hirvonen). *Given* $\varepsilon > 0$ *and* $a \models p$*, there exists* $b \models q$ *such that* $d(a, b) \leq d(p, q) + \varepsilon$

Proof. Fix $\varepsilon > 0$. By the definition of d, there exist realizations $c \models p$ and $c' \models q$ such that $d(c, c') \leq \mathbf{d}(p, q) + \varepsilon$. As $a, c \models p$ then there exists $f \in Aut(\mathbb{M}/A)$ such that $f(c) = a$. Note that $d(a, f(c')) = d(f(c), f(c')) = d(c, c') \leq \mathbf{d}(p, q) + \varepsilon$, where $f(c') \models q$, so $f(c')$ is the required b. $\square_{\text{Fact 1.3.10}}$

Corollary 1.3.11. *Given* $\varepsilon > 0$ *and* $p, q \in$ ga-S(M) *such that* $\mathbf{d}(p, q) < \varepsilon$ *and* $b \models q$*, then there exists* $a_\varepsilon \models p$ *such that* $d(a_\varepsilon, b) < 2\varepsilon$*.*

Proof. By fact 1.3.10, there exists $a_\varepsilon \models p$ such that $d(a_\varepsilon, b) \leq \mathbf{d}(p, q) + \varepsilon$, therefore $d(a_\varepsilon, b) \leq \mathbf{d}(p, q) + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon$. $\square_{\text{Cor. 1.3.11}}$

The previous corollary says that if $d(p, q) < \varepsilon$ and if we fix $b \models q$, we can find a realization $a_\varepsilon \models p$ such that $d(a_\varepsilon, b) < 2\varepsilon$. But we can even find such realization a_ε such that $d(a_\varepsilon, b) < \varepsilon$.

Proposition 1.3.12. *Given* $\varepsilon > 0$ *and* $p, q \in$ ga-S(M) *such that* $\mathbf{d}(p, q) < \varepsilon$ *and* $b \models q$*, then there exists* $a_\varepsilon \models p$ *such that* $d(a_\varepsilon, b) < \varepsilon$

Proof. Since $d(p, q) < \varepsilon$, there exist $a' \models p$ and $b' \models q$ such that $d(a', b') < \varepsilon$ (by definition of distance between types). Since $b', b \models q$, there exists $f \in Aut(\mathbb{M}/M)$ such that $f(b') = b$. Notice that $f(a') \models p$ (because $a' \models p$ and f fixes M pointwise). Since f is an isometry, $d(f(a'), b) = d(f(a'), f(b')) = d(a', b') < \varepsilon$. Notice that the statement of the proposition holds for $a_\varepsilon := f(a')$. $\square_{\text{Prop. 1.3.12}}$

Definition 1.3.13 (Continuity of Types). Let $\mathcal{K}$ be an MAEC and consider $(a_n) \to a$ in $\mathbb{M}$. We say that $\mathcal{K}$ has the *Continuity of Types Property*[2] (for short, *CTP*), if and only if, if ga-tp$(a_n/M) =$ ga-tp(a_0/M) for all $n < \omega$ then ga-tp$(a/M) =$ ga-tp(a_0/M).

Claim 1.3.14. *The distance between types* d *(see Definition 1.3.9) is a pseudo-metric (i.e.,* d *satisfies the triangle inequality and symmetry). Moreover,* d *is a metric (i.e.,* d *is a pesudometric satisfying* $d(p, q) = 0$ *iff* $p = q$*) if and only if* $\mathcal{K}$ *has the CTP.*

Proof.

[2]This property is also called *Perturbation Property in [HH09]*

⇒ Suppose the distance between Galois types defined above is a metric and let $(a_n)_{n<\omega}$ be a convergent sequence such that $(a_n) \to a$ and $\text{ga-tp}(a_0/M) = \text{ga-tp}(a_n/M)$ for all $n < \omega$. Let $\varepsilon > 0$. Since $(a_n) \to a$, there exists $N < \omega$ such that $d(a_n, a) < \varepsilon$ for every $n \geq N$. Therefore,

$$\begin{aligned} d(\text{ga-tp}(a_0/M), \text{ga-tp}(a/M)) &= d(\text{ga-tp}(a_N/M), \text{ga-tp}(a/M)) \\ &< \varepsilon \end{aligned}$$

Since d is a metric, $\text{ga-tp}(a_0/M) = \text{ga-tp}(a/M)$. So, $\mathcal{K}$ has the CTP.

⇐ This claim is proved in [HH09]. But for the sake of completeness, we include a proof. Symmetry and triangle inequality are straightforward to prove. We provide the proof of the fact that $d(p, q) = 0$ iff $p = q$. For the nontrivial direction, assume $d(p, q) = 0$. By definition, given $n < \omega$ there exist $a_n \models p$ and $b_n \models q$ such that $d(a_n, b_n) < \frac{1}{n+1}$. By fact 1.3.10, we may assume $a_0 = a_n$ for every $n < \omega$ (consider a_n and b_n as above, so take $b'_n \models q$ such that $d_(a_0, b'_n) \leq d(p, q) + d(a_n, b_n) < \frac{1}{n+1}$ and consider b'_n instead of b_n). Therefore $(b_n) \to a_0$ and by CTP $p = \text{ga-tp}(a_0/M) = \text{ga-tp}(b_0/M) = q$.

$\square_{\text{Claim 1.3.14}}$

Throughout this document, we also assume CTP (so, distance between Galois types is in fact a metric).

Assuming CTP (i.e., distance between Galois types is a metric), when we require to prove that two Galois types p, q are the same, it is enough to prove that $d(p, q) < \varepsilon$ for every $\varepsilon > 0$. We point out that we can obtain this inequality using suitable instances of ε-splitting (see section 1.4).

Since we can define a metric in $\text{ga-S}(M)$, we define a metric version of stability, as follows:

Definition 1.3.15 (d-stability). Let $\mathcal{K}$ be an MAEC that has a monster model as in remark 1.3.5 and satisfies CTP. We say that $\mathcal{K}$ is μ-d-stable if and only if given any $M \in \mathcal{K}$ of density character μ, $dc(\text{ga-S}(M)) \leq \mu$. If it is clear that we are working in a metric context, we just say that $\mathcal{K}$ is μ-stable.

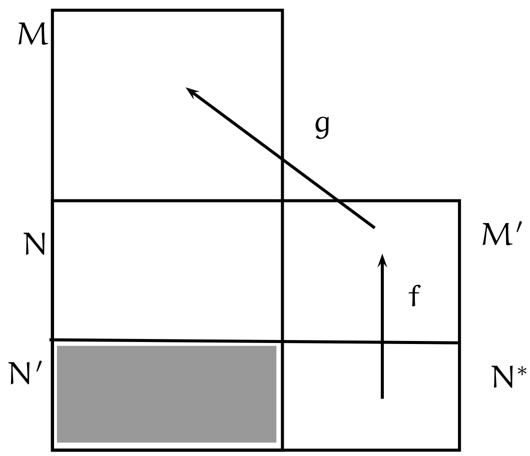
Definition 1.3.16 (d-universality). Let $\mathcal{K}$ be an MAEC and $N \prec_{\mathcal{K}} M$. We say that M is λ-d-*universal* over N iff for every $N' \succ_{\mathcal{K}} N$ with density character λ there exists a $\mathcal{K}$-embedding $f : N' \to M$ such that $f \upharpoonright N = id_N$. We say that M is d-*universal* over N if M is $dc(N)$-universal over N. We drop the prefix d if it is clear that we are working in a metric setting.

Definition 1.3.17 (limit model). Let $\mathcal{K}$ be an MAEC. We say that M is a (μ, θ)-*limit model* if and only if there exist a continuous resolution $\langle M_i : i < \theta \rangle$ of M of length θ such that $dc(M_i) = \mu$ and M_{i+1} is d-universal over M_i for every $i < \theta$.

Remark 1.3.18. Under μ-d-stability, we can prove -as in (discrete) AECs- the existence of μ-d-universal models. We will prove existence of d-universal models in proposition 2.1.1. However, throughout this chapter we assume the existence of universal models.

Proposition 1.3.19. *Let $\mathcal{K}$ be an MAEC with AP. Let $N' \prec_{\mathcal{K}} N \prec_{\mathcal{K}} M$. If M is universal over N, then it is $dc(N)$-universal over N'.*

Proof. Let $N^* \succ_{\mathcal{K}} N'$ be of density character $dc(N)$. By AP and Downward Löhenheim-Skolem axiom, there exist $M' \succ_{\mathcal{K}} N$ of density character $dc(N)$ and a $\prec_{\mathcal{K}}$-embedding $f : N^* \to M$ such that $f \upharpoonright N' = id_{N'}$. Since M is universal over N and $M' \succ_{\mathcal{K}} N$, then there exists a $\prec_{\mathcal{K}}$-embedding $g : M' \to M$ such that $g \upharpoonright N = id_N$. Notice that $g \circ f : N^* \to M$ is a $\prec_{\mathcal{K}}$-embedding which fixes N' pointwise, so M is $dc(N')$—universal over N'.



$\square_{\text{Prop. 1.3.19}}$

Lemma 1.3.20. *Let $\mathcal{K}$ be a MAEC. If $f_i : M_i \to M$ ($i < \mu$) is a $\subseteq$-increasing and continuous (in the metric sense) chain of $\mathcal{K}$-embeddings, then there exists a $\mathcal{K}$-embedding $f : \overline{\bigcup_{i<\mu} M_i} \to M$ which extends $g := \bigcup_{i<\mu} f_i : \bigcup_{i<\mu} M_i \to M$.*

Proof. Let $a \in \overline{\bigcup_{i<\mu} M_i}$, so there exist elements $a_n \in \bigcup_{i<\mu} M_i$ for $n < \omega$, such that $(a_n)_{n<\omega} \to a$. As $(a_n)_{n<\omega}$ is a Cauchy sequence, $(g(a_n))_{n<\omega}$ is also a Cauchy sequence (since g is an isometry). So, there exists $b \in M$ such that $(g(a_n))_{n<\omega} \to b$. Define $f(a) := b$. Proceed in a similar way for every $a \in \overline{\bigcup_{i<\mu} M_i}$. The function f is well-defined: if we take $(a'_n)_{n<\omega}$ a sequence in $\bigcup_{i<\mu} M_i$ such that $(a'_n)_{n<\omega} \to a$, let $b' \in M$ be such that $(g(a'_n))_{n<\omega} \to b'$. We will prove that $b = b'$. Otherwise, let $\varepsilon := d(b, b') > 0$.

Claim 1.3.21. *Given $\varepsilon' > 0$, there exists $N < \omega$ such that for all $n \geq N$ $d(g(a_n), g(a'_n)) < \varepsilon'$.*

Proof. As $(a_n)_{n<\omega} \to a$ and $(a'_n)_{n<\omega} \to a$, there exists $N < \omega$ such that for all $n \geq N$ we have that $d(a_n, a) < \varepsilon'/2$ and $d(a'_n, a) < \varepsilon'/2$, so for all $n \geq N$ we have that $d(a_n, a'_n) \leq d(a_n, a) + d(a, a'_n) < \varepsilon'$. As g is an isometry, for all $n \geq N$ we have that $d(g(a_n), g(a'_n)) < \varepsilon'$.

$\square_{\text{Claim 1.3.21}}$

As $(g(a_n))_{n<\omega} \to b$, $(g(a'_n))_{n<\omega} \to b'$ and by claim 1.3.21, there exists $M < \omega$ such that for all $n \geq M$ we have that $d(g(a_n), b) < \varepsilon/3$, $d(g(a'_n), b') < \varepsilon/3$ and $d(g(a_n), g(a'_n)) < \varepsilon/3$. So, for all $n \geq M$ we have that $d(b, b') \leq d(b, g(a_n)) + d(g(a_n), g(a'_n)) + d(g(a'_n), b') < \varepsilon = d(b, b')$ (contradiction).

Therefore $b = b'$ and so f is well-defined.

We have that f extends g: let $a \in \bigcup_{i<\omega} M_i$, so taking $a_n := a$ $(n < \omega)$ we have that $(a_n)_{n<\omega} \to a$ and $(g(a_n))_{n<\omega}$ is also a constant sequence. So, $f(a) = \lim_{n<\omega} g(a_n) = g(a)$.

Let $c \in f[\overline{\bigcup_{i<\mu} M_i}]$, so there exists $a \in \overline{\bigcup_{i<\mu} M_i}$ such that $f(a) = c$, therefore there exists $(a_n)_{n<\omega}$ a sequence in $\bigcup_{i<\mu} M_i$ such that $(a_n)_{n<\omega} \to a$ and $c := \lim_{n<\omega} g(a_n)$. Therefore $c \in \overline{g[\bigcup_{i<\mu} M_i]} = \overline{\bigcup_{i<\mu} f_i[M_i]}$, so $f[\overline{\bigcup_{i<\mu} M_i}] \subseteq \overline{\bigcup_{i<\mu} f_i[M_i]}$. Take $c \in \overline{\bigcup_{i<\mu} f_i[M_i]} = \overline{g[\bigcup_{i<\mu} M_i]}$, so there exists a sequence $(b_n)_{n<\omega}$ in $\bigcup_{i<\mu} M_i$ such that $(g(b_n))_{n<\omega} \to c$. As $(g(b_n))_{n<\omega}$ is a Cauchy sequence and g is an isometry, we have that $(b_n)_{n<\omega}$ is also a Cauchy sequence. So, there exists $a \in \overline{\bigcup_{i<\mu} M_i}$ such that $(b_n)_{n<\omega} \to a$, and therefore $f(a) = \lim_{n<\omega} g(b_n) = c$, hence $c \in f[\overline{\bigcup_{i<\mu} M_i}]$. So, $f[\overline{\bigcup_{i<\mu} M_i}] = \overline{\bigcup_{i<\mu} f_i[M_i]}$. As $(f_i : i < \mu)$ is a $\subseteq$-increasing and continuous chain of $\mathcal{K}$-embeddings, $f_i[M_i] \prec_{\mathcal{K}} M$, so by axioms 4.(c) and 5. of MAEC we have that $f[\overline{\bigcup_{i<\mu} M_i}] = \overline{\bigcup_{i<\mu} f_i[M_i]} \prec_{\mathcal{K}} M$. Furthermore, for every symbol σ of $L(\mathcal{K})$, f is compatible with the interpretation of σ in $\overline{\bigcup_{i<\mu} M_i}$: f is a limit of $\mathcal{K}$-embeddings – function symbols on these limits are uniquely interpreted by Axiom 4(a), and f being a limit of $\mathcal{K}$-embeddings, distances to interpretations of predicates are preserved. Therefore f is a $\mathcal{K}$-embedding which extends g. $\square_{\text{Lemma 1.3.20}}$

The following lemma is useful for later constructions — usually, it is easier in the metric case to realize *dense* subsets of ga-S(M); the lemma provides a criterion for relative metric Galois saturation.

Lemma 1.3.22. *Suppose that we have an increasing $\prec_{\mathcal{K}}$-chain of models $(N_n : n < \omega)$ such that N_{n+1} realizes a dense subset of* ga-S(N_n)*. Then, every type in* ga-S(N_0) *is realized in* $N_\omega := \overline{\bigcup_{n<\omega} N_n}$.

Proof. Given $p :=$ ga-tp(b/N_0) there exists $q_0 \in$ ga-S(N_0) which is realized in N_1 (by assumption) and $\mathbf{d}(p, q_0) < \frac{1}{2(0+1)^2} = \frac{1}{2}$. Let a_0 be a realization of q_0. By corollary 1.3.11 there exists $b_0 \models p$ such that $d(b_0, a_0) < 2(\frac{1}{2}) = 1$.

The key idea is to build two Cauchy sequences $(a_n)_{n<\omega}$ and $(b_n)_{n<\omega}$ in $\mathbb{M}$ such that $a_n \in N_{n+1}$, ga-tp(b_n/N_0) = ga-tp(b/N_0) for every $n < \omega$ and also a_n and b_n are close enough, so if $c := \lim_{n<\omega} b_n = \lim_{n<\omega} a_n$ then by CTP (Definition 1.3.13) we have that ga-tp(c/N_0) = ga-tp$(b_0/N_0) = p$. Since $c = \lim_{n<\omega} a_n$, then $c \in N_\omega := \overline{\bigcup_{n<\omega} N_n}$, and so p is realized in

N_ω.

The construction: Consider $n > 0$. Since N_{n+1} realizes a dense subset of ga-S(N_n), take $a_n \in N_{n+1}$ a realization of a type $q_n \in$ ga-S(N_n) which satisfies $\mathbf{d}(\text{ga-tp}(b_{n-1}/N_n), q_n) < \frac{1}{2n^2}$. By corollary 1.3.11, take $b_n \models \text{ga-tp}(b_{n-1}/N_n)$ such that $d(b_n, a_n) < 2(\frac{1}{2n^2}) = \frac{1}{n^2}$.

We have that $(a_n)_{n<\omega}$ is a Cauchy sequence: as $b_{n+1} \models \text{ga-tp}(b_n/N_{n+1})$, there exists $g \in Aut(\mathbb{M}/N_{n+1})$ such that $g(b_n) = b_{n+1}$. Since g is an isometry and $a_n \in N_{n+1}$, then $d(b_{n+1}, a_n) = d(g(b_n), g(a_n)) = d(b_n, a_n) < \frac{1}{n^2}$. Therefore, $d(a_{n+1}, a_n) \le d(a_{n+1}, b_{n+1}) + d(b_{n+1}, a_n) < \frac{1}{(n+1)^2} + \frac{1}{n^2} < \frac{2}{n^2}$, so we have that $(a_n : n < \omega)$ is a Cauchy sequence.

Therefore, there exists $c := \lim_{n<\omega} a_n$, $c \in N_\omega$ and also $c = \lim_{n<\omega} b_n$. So, we are done. $\square_{\text{Lemma 1.3.22}}$

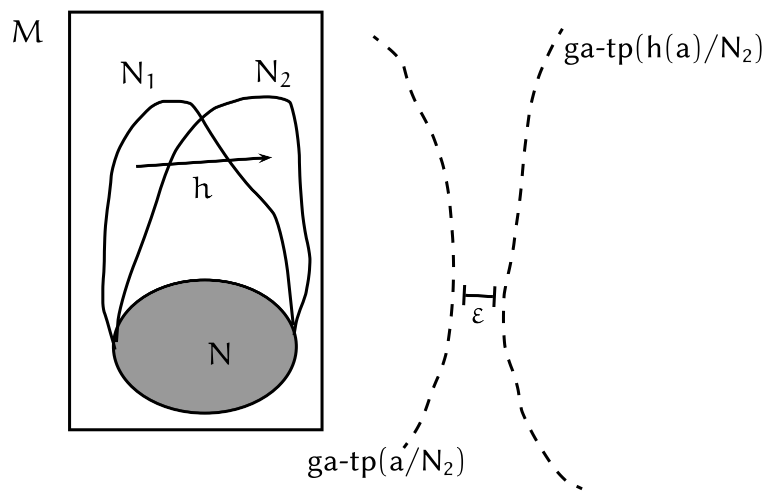
1.4 Smooth independence in MAEC

Assumption 1.4.1. *We summarize the general assumptions which we suppose throughout this section: We assume that $\mathcal{K}$ is an MAEC such that*

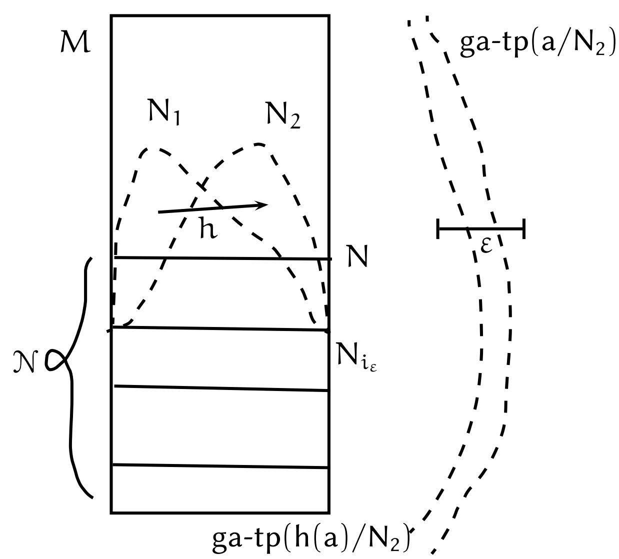
1. *it satisfies AP, JEP, CTP, is μ-d-stable and there exist arbitrarily large enough models.*
2. *so, we can construct a model-homogeneous monster model $\mathbb{M}$ such that every model $M \in \mathcal{K}$ can be $\mathcal{K}$-embedded into $\mathbb{M}$ (Galois types over a model M are orbits under $Aut(\mathbb{M}/M)$)*
3. *Also, every model considered in this section has density character μ -unless otherwise stated-.*

S. Shelah defined a suitable notion of independence called *splitting* (see [She99]) in a general setting of Abstract Elementary Classes, which under stability assumptions is well-behaved; it satisfies monotonicity, extension and stationarity over limit models and local character. But in a general way, we can just work with types over models. Å. Hirvonen and T. Hyttinen gave notions of ε-splitting and 0-independence given in the context of *model-homogeneous metric abstract elementary classes* (see [HH09]). In the homogeneous setting, we can consider Galois types over sets (in the general case, we cannot do that). In this section, we present an adaptation of the notion of ε-splitting and 0-independence given in [HH09], but working with Galois-types over models.

Definition 1.4.2 (ε-splitting and $\downarrow^\varepsilon$). Let $N \prec_{\mathcal{K}} M$ and $\varepsilon > 0$. We say that ga-tp(a/M) ε-*splits* over N iff there exist N_1, N_2 with $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ and $h : N_1 \approx_N N_2$ such that $\mathbf{d}(\text{ga-tp}(a/N_2), h(\text{ga-tp}(a/N_1))) \ge \varepsilon$. We use $a \downarrow^\varepsilon_N M$ to denote the fact that ga-tp(a/M) does not ε-split over N,



Definition 1.4.3. Let $N \prec_{\mathcal{K}} M$. Fix $\mathcal{N} := \langle N_i : i < \sigma \rangle$ a resolution of N. We say that a is *smoothly independent* from M over N relative to $\mathcal{N}$ (denoted by $a \downarrow^{\mathcal{N}}_{N} M$) iff for every $\varepsilon > 0$ there exists $i_\varepsilon < \sigma$ such that $a \downarrow^{\varepsilon}_{N_{i_\varepsilon}} M$.



We called *smooth independence* the notion of independence given above, inspired by [BS91]. In that paper, J. Baldwin and S. Shelah defined *smoothness* as a nice property of an abstract class of models $\mathcal{K}$ which involves increasing chains of models, context where the existence of a kind of monster model holds.

Notation 1.4.4. Let p be a Galois-type over M, N a $\mathcal{K}$-submodel of M and $\mathcal{N}$ a resolution of N. We write $p \downarrow^{\varepsilon}_{N} M$ ($p \downarrow^{\mathcal{N}}_{N} M$) whenever for any realization $a \models p$ we have that $a \downarrow^{\varepsilon}_{N} M$ ($a \downarrow^{\mathcal{N}}_{N} M$).

Proposition 1.4.5 (Invariance of smooth independence). *Let* $N \prec_{\mathcal{K}} M$, $a \in \mathbb{M}$, $\mathcal{N} := \{N_i : i < \sigma\}$ *be a resolution of* N *and* $f \in \mathrm{Aut}(\mathbb{M})$. *Then* $a \downarrow_{N}^{\mathcal{N}} M$ *if and only if* $f(a) \downarrow_{f[N]}^{f[\mathcal{N}]} f[M]$, *where* $f[\mathcal{N}] := \{f[N_i] : i < \sigma\}$.

Proposition 1.4.6 (Monotonicity of smooth independence). *Let* $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$. *Fix* $\mathcal{M}_k := \langle M_i^k : i < \sigma_k \rangle$ *a resolution of* M_k *(*$k = 0, 1$*), where* $\mathcal{M}_0 \subseteq \mathcal{M}_1$. *If* $a \downarrow_{M_0}^{\mathcal{M}_0} M_3$ *then* $a \downarrow_{M_1}^{\mathcal{M}_1} M_2$.

Proof. Let $\varepsilon > 0$. Since $a \downarrow_{M_0}^{\mathcal{M}_0} M_3$, there exists $i_\varepsilon < \sigma_0$ such that $a \downarrow_{M_{i_\varepsilon}^0}^{\varepsilon} M_3$. But $\mathcal{M}_0 \subseteq \mathcal{M}_1$, then there exists $j_\varepsilon < \sigma_1$ such that $M_{i_\varepsilon}^0 = M_{j_\varepsilon}^1$. Therefore, for every h, N_1 and N_2 such that $M_{j_\varepsilon}^1 \prec_{\mathcal{K}} N_1 \overset{h}{\approx}_{M_{j_\varepsilon}^1} N_2 \prec_{\mathcal{K}} M_3$ (in particular for every such $N_1, N_2 \prec_{\mathcal{K}} M_2$) we have that $d(\text{ga-tp}(a/N_2), \text{ga-tp}(h(a)/N_2)) < \varepsilon$. Then $a \downarrow_{M_{j_\varepsilon}^1}^{\varepsilon} M_2$. Since this holds for every $\varepsilon > 0$, then $a \downarrow_{M_1}^{\mathcal{M}_1} M_2$. $\square_{\text{Prop. 1.4.6}}$

Proposition 1.4.7 (Monotonicity of non-ε-splitting). *Let* $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$. *If* $a \downarrow_{M_0}^{\varepsilon} M_3$ *then* $a \downarrow_{M_1}^{\varepsilon} M_2$.

Proof. Suppose $a \downarrow_{M_0}^{\varepsilon} M_3$. If $N_1, N_2 \in \mathcal{K}$ satisfies $M_1 \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M_2$ and $h : N_1 \approx N_2$ fixes M_1 pointwise, in particular $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$ and h fixes M_0 pointwise (because $M_0 \prec_{\mathcal{K}} M_1$). Since $a \downarrow_{M_0}^{\varepsilon} M_3$, then $d(\text{ga-tp}(a/N_2), \text{ga-tp}(h(a)/N_2)) < \varepsilon$. Hence $a \downarrow_{M_1}^{\varepsilon} M_2$. $\square_{\text{Prop. 1.4.7}}$

Although the following claim does not involve uniqueness of independent extensions, we can consider it as a weak version of *stationarity*: every pair of ε-independent extensions are pretty close to one another (in fact, distance between two ε-independent extensions is less than ε).

Lemma 1.4.8 (Weak stationarity). *Suppose that* $N_0 \prec_{\mathcal{K}} N_1 \prec_{\mathcal{K}} N_2$ *and* N_1 *is universal over* N_0. *If* $\text{ga-tp}(a/N_1) = \text{ga-tp}(b/N_1)$, $a \downarrow_{N_0}^{\varepsilon} N_2$ *and* $b \downarrow_{N_0}^{\varepsilon} N_2$, *then* $\mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(b/N_2)) < 2\varepsilon$.

Proof. Since N_1 is universal over N_0, there exists a $\mathcal{K}$-embedding $g : N_2 \to_{N_0} N_1$. So, $N_0 \prec_{\mathcal{K}} g[N_2] \prec_{\mathcal{K}} N_1$.

Since $N_0 \prec_{\mathcal{K}} g[N_2], N_2 \prec_{\mathcal{K}} N_2$, $g^{-1} \restriction g[N_2] : g[N_2] \xrightarrow{\approx}_{N_0} N_2$ and $a \downarrow_{N_0}^{\varepsilon} N_2$, then $\mathbf{d}(\text{ga-tp}(g^{-1}(a)/N_2), \text{ga-tp}(a/N_2)) < \varepsilon$.

Doing a similar argument, it is easy to prove that $\mathbf{d}(\text{ga-tp}(g^{-1}(b)/N_2), \text{ga-tp}(b/N_2)) < \varepsilon$.

Also, since $\text{ga-tp}(a/N_1) = \text{ga-tp}(b/N_1)$ and $g[N_2] \prec_{\mathcal{K}} N_1$, we have $\text{ga-tp}(a/g[N_2]) = \text{ga-tp}(b/g[N_2])$, so $\text{ga-tp}(g^{-1}(a)/N_2) = \text{ga-tp}(g^{-1}(b)/N_2)$.

Therefore,

$$\begin{aligned}\mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(b/N_2)) &\le \mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(g^{-1}(a)/N_2)) \\ &\quad +\mathbf{d}(\text{ga-tp}(g^{-1}(a)/N_2), \text{ga-tp}(g^{-1}(b)/N_2)) \\ &\quad +\mathbf{d}(\text{ga-tp}(g^{-1}(b)/N_2), \text{ga-tp}(b/N_2)) \\ &< \varepsilon + 0 + \varepsilon \\ &= 2\varepsilon\end{aligned}$$

$\square_{\text{Lemma 1.4.8}}$

Proposition 1.4.9 (Extension of $\downarrow^{\mathcal{N}}$ over universal models). *If* $N \prec_{\mathcal{K}} M \prec_{\mathcal{K}} M'$, $\mathcal{N} := \langle N_i : i < \sigma\rangle$ *is a resolution of* N, M *is universal over* N *and* $p := \text{ga-tp}(a/M) \in \text{ga-S}(M)$ *is a Galois type such that* $a \downarrow^{\mathcal{N}}_{N} M$, *then there exists* b *such that* $\text{ga-tp}(b/M) = \text{ga-tp}(a/M)$ *and* $b \downarrow^{\mathcal{N}}_{N} M'$.

Proof. Since M is universal over N, there exists a $\mathcal{K}$-embedding $h' : M' \to_N M$ (recall all models have density character μ, see assumptions at the beginning of section 1.4). Extend h' to an automorphism $h \in Aut(\mathbb{M}/N)$. Since $a \downarrow^{\mathcal{N}}_{N} M$ and $h[M'] \prec_{\mathcal{K}} M$, by monotonicity of $\downarrow^{\mathcal{N}}$ we have that $a \downarrow^{\mathcal{N}}_{N} h[M']$. By invariance, we have that $h^{-1}(a) \downarrow^{\mathcal{N}}_{N} M'$.

Claim 1.4.10. $\text{ga-tp}(a/M) = \text{ga-tp}(h^{-1}(a)/M)$.

Proof. Consider $N_1 := h^{-1}[M]$ and $N_2 := M$. Notice that $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} h^{-1}[M]$ and $h \restriction N_1 : N_1 \approx_N N_2$. Since $a \downarrow^{\mathcal{N}}_{N} M$, by invariance we have that $h^{-1}(a) \downarrow^{\mathcal{N}}_{N} h^{-1}[M]$. So, given $n < \omega$ there exists $i_n < \sigma$ such that $h^{-1}(a) \downarrow^{\frac{1}{n+1}}_{N_{i_n}} h^{-1}[M]$.

By monotonicity of non-ε-splitting (Proposition 1.4.7), we may conclude that $h^{-1}(a) \downarrow^{\frac{1}{n+1}}_{N} h^{-1}[M]$ for every $n < \omega$.

Since $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} h^{-1}[M]$, we therefore have that for every $n < \omega$ $\mathbf{d}(\text{ga-tp}(h^{-1}(a)/N_2), \text{ga-tp}((h \circ h^{-1})(a)/N_2)) < \frac{1}{n+1}$, taking h as the witness for non-$\frac{1}{n+1}$-splitting.

Since $N_2 = M$, we may conclude that $\text{ga-tp}(a/M) = \text{ga-tp}(h^{-1}(a)/M)$. $\square_{\text{Claim 1.4.10}}$

Since $\text{ga-tp}(a/M) = \text{ga-tp}(h^{-1}(a)/M)$ (by Claim 1.4.10), there exists $g \in Aut(\mathbb{M}/M)$ such that $g(h^{-1}(a)) = a$. Recall that $h^{-1}(a) \downarrow^{\mathcal{N}}_{N} M'$, so by invariance we have that $g(h^{-1}(a)) \downarrow^{\mathcal{N}}_{N} g[M']$, i.e.: $a \downarrow^{\mathcal{N}}_{N} g[M']$. Applying invariance again, we have that $g^{-1}(a) \downarrow^{\mathcal{N}}_{N} M'$. Take $b := g^{-1}(a)$. $\square_{\text{Prop. 1.4.9}}$

Proposition 1.4.11 (stationarity (1)). *If* $N \prec_{\mathcal{K}} M \prec_{\mathcal{K}} M'$, M *is universal over* N, $\mathcal{N} := \langle N_i : i < \sigma\rangle$ *a resolution of* N *and* $p := \text{ga-tp}(a/M) \in \text{ga-S}(M)$ *is a Galois type such that* $a \downarrow^{\mathcal{N}}_{N} M$, *then there exists a unique extension* $p^* \supset p$ *over* M' *which is independent (relative to* $\mathcal{N}$*) from* M' *over* N.

Proof. By proposition 1.4.9, there exists at least one extension $p^* := \text{ga-tp}(b/M')$ of p such that

$$b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathcal{N}}_{N} M'$$

Let $q^* := \text{ga-tp}(c/M') \supset p$ be any other extension of p such that for any $c \models q^*$, $c \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathcal{N}}_{N} M'$. So, $p^* \restriction M = q^* \restriction M = p$, $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathcal{N}}_{N} M'$ and $c \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathcal{N}}_{N} M'$. We will prove that $p^* = q^*$.

Let $\varepsilon > 0$. So, there exist $i^a_\varepsilon, i^b_\varepsilon < \sigma$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{N_{i^a_\varepsilon}} M'$ and $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{N_{i^b_\varepsilon}} M'$. Taking $i := \max\{i^a_\varepsilon, i^b_\varepsilon\}$, by monotonicity of non-ε-splitting we have that $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{N_i} M'$ and $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{N_i} M'$.

Since M is universal over N_i (because M is universal over N), $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{N_i} M'$, $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{N_i} M'$ and $p^* \restriction M = q^* \restriction M$, by lemma 1.4.8 we have that $\mathbf{d}(p^*, q^*) < 2\varepsilon$. Therefore $p^* = q^*$, as ε was taken arbitrary. $\square_{\text{Prop. 1.4.11}}$

The following property of smooth independence (called *antireflexivity*) is the metric version of the following property of thorn-forking in the first order setting: if $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{þ}_{B} a$ then $a \in acl(B)$.

Proposition 1.4.12 (Antireflexivity). *Let $M \prec_{\mathcal{K}} N$ where M is a (μ, θ)-limit model witnessed by $\mathcal{M} := \{M_i : i < \theta\}$. If $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathcal{M}}_{M} N$ and $a \in N$, then $a \in M$.*

Proof. Let $\varepsilon > 0$ and $i_\varepsilon < \theta$ be such that $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{M_{i_\varepsilon}} N$. Since M is universal over M_{i_ε}, there exists an $\prec_{\mathcal{K}}$-embedding $f : N \to M$ which fixes M_{i_ε} pointwise. Define $c := f(a)$. Notice that $c \in M$. Setting $N_1 := N$ and $N_2 := f[N]$, notice that $M_{i_\varepsilon} \prec_{\mathcal{K}} N_1 \overset{f}{\approx}_{M_{i_\varepsilon}} N_2 \prec_{\mathcal{K}} N$. Since $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{M_{i_\varepsilon}} N$, then

$$\begin{aligned} d(\text{ga-tp}(a/N_2), \text{ga-tp}(c/N_2)) &= d(\text{ga-tp}(a/N_2), \text{ga-tp}(f(a)/N_2)) \\ &< \varepsilon \end{aligned}$$

Since $c = f(a) \in f[N] = N_2$, c is the unique realization of $\text{ga-tp}(c/N_2)$. Therefore, we can find $a' \models \text{ga-tp}(a/N_2)$ such that $d(a', c) < \varepsilon$ (by definition of distance between types).

Since $a' \models \text{ga-tp}(a/N_2)$, there exists $g \in Aut(\mathbb{M}/N_2)$ such that $g(a) = a'$. Therefore,

$$\begin{aligned} d(a, c) &= d(g(a), g(c)) \text{ (g is an isometry)} \\ &= d(a', c) \text{ (since } c \in N_2 \text{ and } g \in Aut(\mathbb{M}/N_2)) \\ &< \varepsilon \end{aligned}$$

Defining $B(a, \varepsilon) := \{b \in N : d(a, b) < \varepsilon\}$, we have that $c \in B(a, \varepsilon) \cap f[N] \subseteq B(a, \varepsilon) \cap M$, so $B(a, \varepsilon) \cap M \neq \emptyset$, hence $a \in \overline{M} = M$. $\square_{\text{Prop. 1.4.12}}$

The following fact strongly uses μ-d-stability.

Proposition 1.4.13 (Local character of ε-non-splitting). *Let $\varepsilon > 0$. For every $p \in \text{ga-S}(N)$ with N of density character $> \mu$ there exists $M \prec_{\mathcal{K}} N$ with density character μ such that $p \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\varepsilon}_{M} N$*

Proof. Suppose that there exists some $p := \text{ga-tp}(\overline{a}/N)$ such that $p \not\downarrow^{\varepsilon}_{M} N$ for every $M \prec_{\mathcal{K}} N$ with density character μ. If $\overline{a} \in N$, it is straightforward to see that p does not ε-split over its domain. Then, suppose that $\overline{a} \notin N$.

Define $\chi := \min\{\kappa : 2^\kappa > \mu\}$. So, $\chi \leq \mu$ and $2^{<\chi} \leq \mu$.

We will construct a sequence of models $\langle M_\alpha, N_{\alpha,1}, N_{\alpha,2} : \alpha < \chi\rangle$ in the following way: First, take $M_0 \prec_{\mathcal{K}} N$ as any submodel of density character μ.

Suppose $\alpha := \gamma + 1$ and that M_γ (with density character μ) has been constructed. Then p ε-splits over M_γ. Then there exist $M_\gamma \prec_{\mathcal{K}} N_{\gamma,1}, N_{\gamma,2} \prec_{\mathcal{K}} N$ with density character μ and $F_\gamma : N_{\gamma,1} \approx_{M_\gamma} N_{\gamma,2}$ such that $d(F_\gamma(p \restriction N_{\gamma,1}), p \restriction N_{\gamma,2}) \geq \varepsilon$. Take $M_{\gamma+1} \prec_{\mathcal{K}} N$ a submodel of size μ which contains $|N_{\gamma,1}| \cup |N_{\gamma,2}|$. At limit stages α, take $M_\alpha := \overline{\bigcup_{\gamma<\alpha} M_\gamma}$.

Let us construct a sequence $\langle M^*_\alpha : \alpha \leq \chi\rangle$ of models and a tree $\langle h_\eta : \eta < \alpha\rangle$ ($\alpha \leq \chi$) of $\mathcal{K}$-embeddings such that:

1. $\gamma < \alpha$ implies $M^*_\gamma \prec_{\mathcal{K}} M^*_\alpha$.
2. $M^*_\alpha := \overline{\bigcup_{\gamma<\alpha} M^*_\gamma}$ if α is limit.
3. $\gamma < \alpha$ and $\eta \in {}^{\alpha}2$ imply that $h_{\eta\restriction\gamma} \subset h_\eta$.
4. $h_\eta : M_\alpha \to M^*_\alpha$ for every $\eta \in {}^{\alpha}2$.
5. If $\eta \in {}^{\gamma}2$ then $h_{\eta^\frown 0}(N_{\gamma,1}) = h_{\gamma^\frown 1}(N_{\gamma,2})$

Take $M^*_0 := M_0$ and $h_{\langle\rangle} := id_{M_0}$.

If α is limit, take $M^*_\alpha := \overline{\bigcup_{\gamma<\alpha} M^*_\gamma}$ and if $\eta \in {}^{\alpha}2$ define $h_\eta := \overline{\bigcup_{\gamma<\alpha} h_{\eta\restriction\gamma}}$, the unique extension of $\bigcup_{\gamma<\alpha} h_{\eta\restriction\gamma}$ to $M_\alpha = \overline{\bigcup_{\gamma<\alpha} M_\gamma}$.

If $\alpha := \gamma + 1$, let $\eta \in {}^{\gamma}2$. Take $\overline{h_\eta} \supset h_\eta$ any automorphism of the monster model $\mathbb{M}$ (this is possible because $\mathbb{M}$ is model-homogeneous).

Notice that $\overline{h_\eta} \circ F_\gamma(N_{\gamma,1}) = \overline{h_\eta}(N_{\gamma,2})$. Define $h_{\eta^\frown 0}$ as any extension of $\overline{h_\eta} \circ F_\gamma$ to $M_{\gamma+1}$ and $h_{\eta^\frown 1}$ as $\overline{h_\eta} \restriction M_{\gamma+1}$. Take $M^*_{\gamma+1} \prec_{\mathcal{K}} N$ as any model with density character μ which contains $h_{\eta^\frown l}(M_{\gamma+1})$ for any $\eta \in {}^{\gamma}2$ and $l = 0, 1$.

Now, for every $\eta \leq {}^{\chi}2$, let H_η be an automorphism of $\mathbb{M}$ which extends h_η,

Claim 1.4.14. *If* $\eta \neq \nu \in {}^{\chi}2$ *then* $d(\text{ga-tp}(H_\eta(\overline{a})/M^*_\chi), \text{ga-tp}(H_\nu(\overline{a})/M^*_\chi)) \geq \varepsilon$.

Proof. Suppose not, then $d(\text{ga-tp}(H_\eta(\bar{a})/M^*_\chi), \text{ga-tp}(H_\nu(\bar{a})/M^*_\chi)) < \varepsilon$. Let $\rho := \eta \wedge \nu$. Without loss of generality, suppose that $\rho^\frown 0 \leq \eta$ and $\rho^\frown 1 \leq \nu$. Let $\gamma := lg(\rho)$. Since $h_{\rho^\frown 0}(N_{\gamma,1}) = h_{\rho^\frown 1}(N_{\gamma,2}) \prec_{\mathcal{K}} M^*_\chi$, then $d(\text{ga-tp}(H_\eta(\bar{a})/h_{\rho^\frown 0}(N_{\gamma,1})), \text{ga-tp}(H_\nu(\bar{a})/h_{\rho^\frown 1}(N_{\gamma,2})) < \varepsilon$. Also[3]

$$\begin{aligned} d(\text{ga-tp}(H_\nu^{-1} \circ H_\eta(\bar{a})/F_\gamma(N_{\gamma,1})), \text{ga-tp}(\bar{a}/N_{\gamma,2})) &= \\ d(\text{ga-tp}(H_\eta(\bar{a})/h_{\rho^\frown 0}(N_{\gamma,1})), \text{ga-tp}(H_\nu(\bar{a})/h_{\rho^\frown 1}(N_{\gamma,2})) &< \varepsilon \end{aligned}$$

(as H_ν is an isometry, $h_{\rho^\frown 0} = h_\rho \circ F_\gamma$, $\rho < \nu$, $\rho^\frown 0 \leq \eta$ and $\rho^\frown 1 \leq \nu$). Since $H_\nu^{-1} \circ H_\eta \supset F_\gamma$, then $d(F_\gamma(p \restriction N_{\gamma,1}), p \restriction N_{\gamma,2}) < \varepsilon$, which contradicts the choice of $N_{\gamma,1}$, $N_{\gamma,2}$ and F_γ. This finishes the proof of claim 1.4.14 $\square_{\text{Claim 1.4.14}}$

We have that $dc(M^*_\chi) = \mu$, but claim 1.4.14 says that there are at least $2^\chi > \mu$ many types mutually at distance at least ε. Therefore $dc(\text{ga-S}(M^*_\chi)) > \mu$, which contradicts μ-d-stability. $\square_{\text{Prop. 1.4.13}}$

Corollary 1.4.15 (Existence). *For every $\bar{a} \in \mathbb{M}$ and every $N \in \mathcal{K}$ there exist $M \prec_{\mathcal{K}} N$ with density character μ and a resolution $\mathcal{M} := \langle M_i : i < \omega\rangle$ of M such that $a \perp^{\mathcal{M}}_M N$.*

Proof. Let $n < \omega$. By proposition 1.4.13, there exists $M_n \prec_{\mathcal{K}} N$ with density character μ such that $\bar{a} \perp^{\frac{1}{n+1}}_{M_n} N$. By monotonicity, without loss of generality we can assume that $m < n < \omega$ implies $M_m \prec_{\mathcal{K}} M_n$. Take $M := \overline{\bigcup_{n<\omega} M_n}$. Notice that $dc(M) = \mu$.
It is straightforward to see that $\bar{a} \perp^{\mathcal{M}}_M N$. $\square_{\text{Prop. 1.4.15}}$

Lemma 1.4.16 (Continuity of smooth independence). *Let $(b_n)_{n<\omega}$ be a convergent sequence and $b := \lim_{n<\omega} b_n$. If $b_n \perp^{\mathcal{N}}_N M$ for every $n < \omega$, then $b \perp^{\mathcal{N}}_N M$.*

Proof. Since $b_n \perp^{\mathcal{N}}_N M$ for every $n < \omega$, for an arbitrary fixed $\varepsilon > 0$ there exists $i_{n,\varepsilon} < \sigma$ such that for every $N_{i_{n,\varepsilon}} \prec_{\mathcal{K}} N^1 \overset{h}{\approx}_{N_{i_{n,\varepsilon}}} N^2 \prec_{\mathcal{K}} M$ we have that $d(\text{ga-tp}(b_n/N^2), \text{ga-tp}(h(b_n)/N^2)) < \varepsilon/3$.

Let $K < \omega$ be such that for every $n \geq K$ we have that $d(b_n, b) < \varepsilon/3$. Thus, $d(\text{ga-tp}(b_n/N^2), \text{ga-tp}(b/N^2)) < \varepsilon/3$ for every $n \geq K$.

Since h is an isometry, we have that $(h(b_n)) \to h(b)$ and also for every $n \geq K$ we have that $d(h(b_n), h(b)) < \varepsilon/3$ (and therefore $d(\text{ga-tp}(h(b_n)/N^2), \text{ga-tp}(h(b)/N^2)) < \varepsilon/3$).

Hence, for any $n \geq K$ we have that

$$\begin{aligned} d(\text{ga-tp}(h(b)/N^2), \text{ga-tp}(b/N^2)) &\leq d(\text{ga-tp}(h(b)/N^2), \text{ga-tp}(h(b_n)/N^2)) + \\ &\quad d(\text{ga-tp}(h(b_n)/N^2), \text{ga-tp}(b_n/N^2)) + \\ &\quad d(\text{ga-tp}(b_n/N^2), \text{ga-tp}(b/N^2)) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

[3]This distance between Galois types makes sense, as $h_{\rho^\frown 0}(N_{\gamma,1}) = h_{\rho^\frown 1}(N_{\gamma,2})$.

Therefore, $b\downarrow^{\varepsilon}_{N_{i_n,\varepsilon}} M$ and so, $b\downarrow^{\mathcal{N}}_{N} M$. $\square_{\text{Lemma 1.4.16}}$

Proposition 1.4.17 (stationarity (2)). *Let* $M_0 \prec_{\mathcal{K}} M \prec_{\mathcal{K}} N$ *be such that* M *is a* (μ,σ)*-limit model over* M_0*, witnessed by* $\mathcal{M} := \langle M_i : i < \sigma\rangle$*. If* $a, b\downarrow^{\mathcal{M}}_{M} N$ *and* $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$*, then* $\text{ga-tp}(a/N) = \text{ga-tp}(b/N)$.

Proof. Let $\varepsilon > 0$. Since $a, b\downarrow^{\mathcal{M}}_{M} N$, there exists $i < \sigma$ such that $a, b\downarrow^{\varepsilon}_{M_i} N$ (by definition and monotonicity of non-ε-splitting). Since M_{i+1} is universal over M_i and $M_i \prec_{\mathcal{K}} N$, there exists a $\prec_{\mathcal{K}}$-embedding $f : N \to_{M_i} M_{i+1}$. Also, since $M_i \prec_{\mathcal{K}} f[N] \overset{f^{-1}}{\approx}_{M_i} N \prec_{\mathcal{K}} N$ and $a\downarrow^{\varepsilon}_{M_i} N$, then $d(\text{ga-tp}(a/N), \text{ga-tp}(f^{-1}(a)/N)) < \varepsilon$. Doing a similar argument, we have that $d(\text{ga-tp}(b/N), \text{ga-tp}(f^{-1}(b)/N)) < \varepsilon$.

On the other hand, we have that $\text{ga-tp}(a/f[N]) = \text{ga-tp}(b/f[N])$ (since by hypothesis $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$ and $f[N] \prec_{\mathcal{K}} M_{i+1} \prec_{\mathcal{K}} M$), therefore we have that $\text{ga-tp}(f^{-1}(a)/N) = \text{ga-tp}(f^{-1}(b)/N)$.

Hence

$$\begin{aligned} d(\text{ga-tp}(a/N), \text{ga-tp}(b/N)) &\leq \text{ga-tp}(a/N) + \text{ga-tp}(f^{-1}(a)/N) \\ &+ d(\text{ga-tp}(f^{-1}(a)/N), \text{ga-tp}(f^{-1}(b)/N)) \\ &+ d(\text{ga-tp}(f^{-1}(b)/N), \text{ga-tp}(b/N)) \\ &< \varepsilon + 0 + \varepsilon \\ &= 2\varepsilon \end{aligned}$$

Therefore, $\text{ga-tp}(a/N) = \text{ga-tp}(b/N)$. $\square_{\text{Prop. 1.4.17}}$

Remark 1.4.18. Notice that stationarity (1) (proposition 1.4.11) and stationarity (2) (proposition 1.4.17) are a bit different. The first one states uniqueness of smooth independent extensions in the case that $N \prec_{\mathcal{K}} M \prec_{\mathcal{K}} M'$, M is universal over N, $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$ and $a, b\downarrow^{\mathcal{N}}_{N} M$; the second one is a bit weaker in the sense that we just require $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$ and $a, b\downarrow^{\mathcal{M}}_{M} M'$ if $M \prec_{\mathcal{K}} M'$, where M still keeps a kind of universality. We proved this other version of stationarity because we need to use it to prove an instance of transitivity of smooth independence.

Proposition 1.4.19 (transitivity). *Let* $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2$ *be such that* M_0 *is a* (μ,σ_0)*-limit model over some* $M' \prec_{\mathcal{K}} M_0 \prec_{\mathcal{K}} M_1$ *(witnessed by* $\mathcal{M}_0$*) and* M_1 *is a* (μ,σ_1)*-limit model over* M_0 *(witnessed by* $\mathcal{M}'_1$*). Let* $\mathcal{M}_1 := \mathcal{M}_0 \cup \mathcal{M}'_1$*, so* $\mathcal{M}_0 \subset \mathcal{M}_1$*. Then* $a\downarrow^{\mathcal{M}_0}_{M_0} M_2$ *iff* $a\downarrow^{\mathcal{M}_0}_{M_0} M_1$ *and* $a\downarrow^{\mathcal{M}_1}_{M_1} M_2$.

Proof. ($\Rightarrow$) By monotonicity.

($\Leftarrow$) Suppose $a\downarrow^{\mathcal{M}_0}_{M_0} M_1$ and $a\downarrow^{\mathcal{M}_1}_{M_1} M_2$. Notice that M_1 is universal over M_0. Therefore, by extension property (proposition 1.4.9), there exists $b \models \text{ga-tp}(a/M_1)$ such that $b\downarrow^{\mathcal{M}_0}_{M_0} M_2$. By monotonicity, we have that $b \downarrow^{\mathcal{M}_1}_{M_1} M_2$. Since $a, b \downarrow^{\mathcal{M}_1}_{M_1} M_2$, $\text{ga-tp}(a/M_1) = \text{ga-tp}(b/M_1)$ and since M_1 is a (μ, σ)-limit model over M_0, then by stationarity (2) (proposition 1.4.17) we have that $\text{ga-tp}(a/M_2) = \text{ga-tp}(b/M_2)$. Since $b \downarrow^{\mathcal{M}_0}_{M_0} M_2$, then $a \downarrow^{\mathcal{M}_0}_{M_0} M_2$. $\square_{\text{Prop. 1.4.19}}$

1.5 Another independence notion in d-tame metric abstract elementary classes.

Tame AECs are a very special kind of AECs which have a categoricity transfer theorem (see [GV06a]) and a nice stability transfer theorem (see [BKV06]). In fact -under $\aleph_0$-tameness and $\aleph_0$-locality (assuming $LS(\mathcal{K}) = \aleph_0$)-, J. Baldwin, D. Kueker and M. VanDieren proved in [BKV06] that $\aleph_0$-Galois-stability implies κ-Galois-stability for every cardinality κ. First, they proved that $\aleph_0$-Galois-stability implies $\aleph_n$-Galois stability for every $n < \omega$ (in fact, their argument works for getting κ-Galois-stability if $cf(\kappa) > \omega$) and so (by $\aleph_0$-locality) $\aleph_\omega$-Galois-stable (where the same argument works for getting κ-Galois stability if $cf(\kappa) = \omega$).

In this section, we provide a definition of tameness adapted to the setting of metric abstract elementary classes and a suitable notion of independence in that setting, which we will use in chapter 4 for proving the following upward stability transfer theorem:

Theorem 4.0.7. *Let $\mathcal{K}$ be a μ-**d**-tame (for some $\mu < \kappa$) MAEC. Suppose that $\mathcal{K}$ is $[LS(\mathcal{K}), \kappa)$-cofinally d-stable. If $cf(\kappa) \geq \zeta^*$ then $\mathcal{K}$ is κ-**d**-stable.*

We will prove theorem 4.0.7 in chapter 4.

Definition 1.5.1. Let $\mathcal{K}$ be an MAEC. We say that $\mathcal{K}$ is $[\zeta, \kappa)$-*cofinally d-stable* if and only if for any cardinal θ such that $\zeta \leq \theta < \kappa$ there exists $\theta \leq \theta' < \kappa$ such that $\mathcal{K}$ is θ'-d-stable.

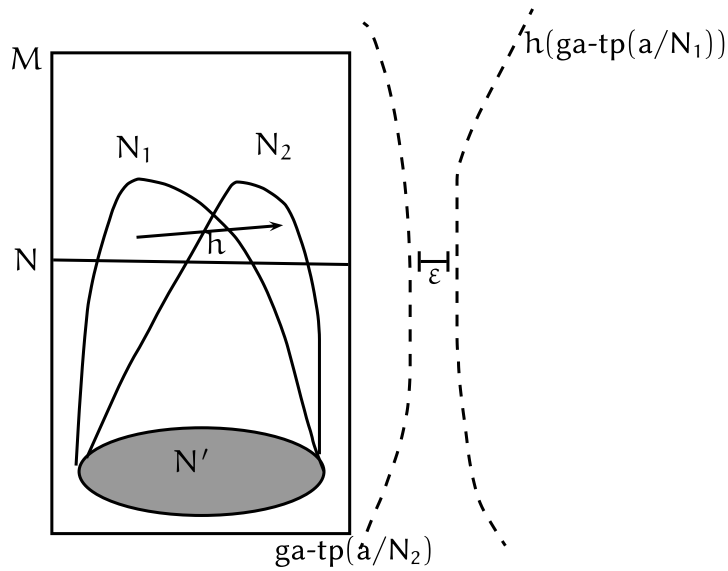
Roughly speaking, a (discrete) AEC is μ-*tame* if and only if the difference between two Galois types over a model $M \in \mathcal{K}$ is determined by a $\mathcal{K}$-submodel of M of cardinality μ. Since we have defined a distance between Galois types in MAECs, difference between two Galois types p and q means that $d(p, q) > 0$. Using this characterization of difference of Galois types via the distance between two Galois types, we adapt the notion of tameness in the metric context.

Definition 1.5.2 (**d**-tameness). Let $\mathcal{K}$ be a MAEC and $\mu \geq LS(\mathcal{K})$. We say that $\mathcal{K}$ is μ-**d**-*tame* iff for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if for any $M \in \mathcal{K}$ of density character $\geq \mu$ if we have that $\mathbf{d}(p, q) \geq \varepsilon$ where $p, q \in \text{ga-S}(M)$, then there exists $N \prec_{\mathcal{K}} M$ of density character μ such that $\mathbf{d}(p \upharpoonright N, q \upharpoonright N) \geq \delta_\varepsilon$.

Assumption 1.5.3. *Throughout this section, we assume that $\mathcal{K}$ is a μ-d-tame and a λ-d-stable MAEC.*

Notation 1.5.4. Let $\kappa > \mu$. Define $\lambda := \min\{\theta < \kappa : \mu < \theta$ and $\mathcal{K}$ is θ-**d**-stable $\}$ (if it makes sense), $\zeta := \min\{\xi : 2^\xi > \lambda\}$ and $\zeta^* := \max\{\mu^+, \zeta\}$.

Definition 1.5.5 (tame ε-splitting). Let $N \prec_{\mathcal{K}} M$ and $\varepsilon > 0$. We say that ga-tp(a/M) ζ^*-*tamely* ε*-splits* over N iff for every submodel $N' \prec_{\mathcal{K}} N$ with density character $< \zeta^*$, there are models $N' \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ with density character $< \zeta^*$ and $h : N_1 \approx_{N'} N_2$ such that $\mathbf{d}(\text{ga-tp}(a/N_2), h(\text{ga-tp}(a/N_1))) \geq \varepsilon$. If ga-tp$(a/M)$ does not ζ^*-tamely ε-split over N, we denote that by $a \downarrow_N^{T,\varepsilon} M$.



Definition 1.5.6. Let $N \prec_{\mathcal{K}} M$. We say that a is *tamely independent* from M over $\mathcal{N}$ iff for every $\varepsilon > 0$ we have that $a \downarrow_N^{T,\varepsilon} M$. We denote this by $a \downarrow_N^{T} M$

Proposition 1.5.7 (Monotonicity). *Let* $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$ *and suppose that* $a \downarrow_{M_0}^{T} M_3$. *Then* $a \downarrow_{M_1}^{T} M_2$.

Proof. Since $a \downarrow_{M_0}^{T} M_3$, given $\varepsilon > 0$ there exists a model $N' \prec_{\mathcal{K}} M_0$ with density character $< \zeta^*$ such that for every pair of models $N' \prec_{\mathcal{K}} N_1 \overset{h}{\approx}_{N'} N_2 \prec_{\mathcal{K}} M_3$ with density character $< \zeta^*$ we have that $d(\text{ga-tp}(a/N_2), \text{ga-tp}(h(a)/N_2)) < \varepsilon$. But we have that $N' \prec_{\mathcal{K}} M_1$ and also in particular if $N' \prec_{\mathcal{K}} N_1 \overset{h}{\approx}_{N'} N_2 \prec_{\mathcal{K}} M_2$. Therefore, $a \downarrow_{M_1}^{T} M_2$. $\square_{\text{Prop. 1.5.7}}$

Fact 1.5.8 (Invariance). *Let* $f \in Aut(\mathbb{M})$. *If* $a \downarrow_N^{T,\varepsilon} M$ *then* $f(a) \downarrow_{f(N)}^{T,\varepsilon} f(M)$.

Recall that we are assuming that $\mathcal{K}$ is λ-d-stable (see assumption 1.5.3). Local character of ζ^*-tamely ε-non-splitting follows from λ-d-stability.

Lemma 1.5.9 (Local character of ζ^*-tamely ε-non-splitting). *For every* M, a *and every* $\varepsilon > 0$ *there exists* $N \prec_{\mathcal{K}} M$ *of density character* $< \zeta^*$ *such that* $a \downarrow_N^{T,\varepsilon} M$.

Proof. Suppose that there exists $p := \text{ga-tp}(\bar{a}/N)$ such that $p \not\downarrow^{T,\varepsilon}_{M} N$ for every $M \prec_{\mathcal{K}} N$ with density character $< \zeta^*$. If $\bar{a} \in N$, it is straightforward to see that p does not ε-split over its domain. Then, suppose that $\bar{a} \notin N$.

We will construct a sequence of models $\langle M_\alpha, N_{\alpha,1}, N_{\alpha,2} : \alpha < \zeta\rangle$in the following way: First, take $M_0 \prec_{\mathcal{K}} N$ as any submodel of density character $< \zeta^*$.

Suppose $\alpha := \gamma + 1$ and that M_γ (with density character $< \zeta^*$) has been constructed. Therefore p ε-splits over M_γ. Then there exist $M_\gamma \prec_{\mathcal{K}} N_{\gamma,1}, N_{\gamma,2} \prec_{\mathcal{K}} N$ with density character $< \zeta^*$ and $F_\gamma : N_{\gamma,1} \approx_{M_\gamma} N_{\gamma,2}$ such that $d(F_\gamma(p \restriction N_{\gamma,1}), p \restriction N_{\gamma,2}) \geq \varepsilon$. Take $M_{\gamma+1} \prec_{\mathcal{K}} N$ a submodel of size $< \zeta^*$ which contains $|N_{\gamma,1}| \cup |N_{\gamma,2}|$. At limit stages α, take $M_\alpha := \overline{\bigcup_{\gamma<\alpha} M_\gamma}$.

Remark 1.5.10. Notice that $\langle M_\gamma : \gamma < \zeta\rangle$ is a $\prec_{\mathcal{K}}$-increasing and continuous sequence such that $a \not\downarrow^{T,\varepsilon}_{M_\gamma} M_{\gamma+1}$ for every $\gamma < \zeta$.

Let us construct a sequence $\langle M^*_\alpha : \alpha \leq \zeta\rangle$ of models and a tree $\langle h_\eta : \eta \in {}^{\alpha}2\rangle$ $(\alpha \leq \zeta)$ of $\mathcal{K}$-embeddings such that:

1. $\gamma < \alpha$ implies $M^*_\gamma \prec_{\mathcal{K}} M^*_\alpha$.
2. $M^*_\alpha := \overline{\bigcup_{\gamma<\alpha} M^*_\gamma}$ if α is limit.
3. $\gamma < \alpha$ and $\eta \in {}^{\alpha}2$ imply that $h_{\eta\restriction\gamma} \subset h_\eta$.
4. $h_\eta : M_\alpha \to M^*_\alpha$ for every $\eta \in {}^{\alpha}2$.
5. If $\eta \in {}^{\gamma}2$ then $h_{\eta^\frown 0}(N_{\gamma,1}) = h_{\gamma^\frown 1}(N_{\gamma,2})$

Take $M^*_0 := M_0$ and $h_{\langle\rangle} := id_{M_0}$.

If α is limit, take $M^*_\alpha := \overline{\bigcup_{\gamma<\alpha} M^*_\gamma}$ and if $\eta \in {}^{\alpha}2$ define $h_\eta := \overline{\bigcup_{\gamma<\alpha} h_{\eta\restriction\gamma}}$.

If $\alpha := \gamma + 1$, let $\eta \in {}^{\gamma}2$. Take $\overline{h_\eta} \supset h_\eta$ any automorphism of the monster model $\mathbb{M}$ (this is possible because $\mathbb{M}$ is model-homogeneous).

Notice that $\overline{h_\eta} \circ F_\gamma(N_{\gamma,1}) = \overline{h_\eta}(N_{\gamma,2})$. Define $h_{\eta^\frown 0}$ as any extension of $\overline{h_\eta} \circ F_\gamma$ to $M_{\gamma+1}$ and $h_{\eta^\frown 1}$ as $\overline{h_\eta} \restriction M_{\gamma+1}$. Take $M^*_{\gamma+1} \prec_{\mathcal{K}} N$ as any model with density character $< \zeta^*$ which contains $h_{\eta^\frown l}(M_{\gamma+1})$ for any $\eta \in {}^{\gamma}2$ and $l = 0, 1$.

Take H_η an automorphism of $\mathbb{M}$ which extends h_η, for every $\eta \leq {}^{\zeta}2$.

Claim 1.5.11. *If* $\eta \neq \nu \in {}^{\zeta}2$ *then* $d(\text{ga-tp}(H_\eta(\bar{a})/M^*_\zeta), \text{ga-tp}(H_\nu(\bar{a})/M^*_\zeta)) \geq \varepsilon$.

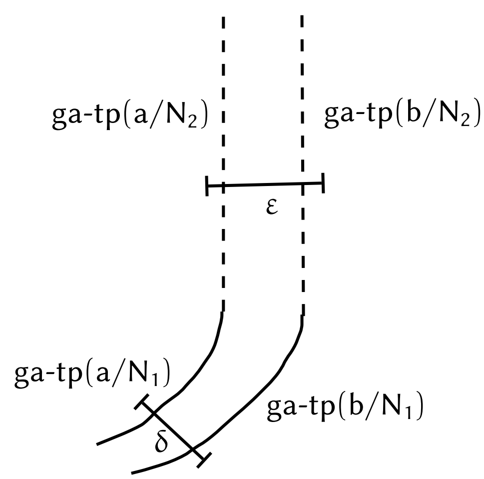
Proof. Suppose not, then $d(\text{ga-tp}(H_\eta(\bar{a})/M^*_\zeta), \text{ga-tp}(H_\nu(\bar{a})/M^*_\zeta)) < \varepsilon$. Let $\rho := \eta \wedge \nu$. Without loss of generality, suppose that $\rho^\frown 0 \leq \eta$ and $\rho^\frown 1 \leq \nu$. Let $\gamma := \mathrm{length}(\rho)$. Since $h_{\rho^\frown 0}(N_{\gamma,1}) = h_{\rho^\frown 1}(N_{\gamma,2}) \prec_{\mathcal{K}} M^*_\zeta$, therefore $d(\text{ga-tp}(H_\eta(\bar{a})/h_{\rho^\frown 0}(N_{\gamma,1})), \text{ga-tp}(H_\nu(\bar{a})/h_{\rho^\frown 1}(N_{\gamma,2})) < \varepsilon$. Also

$$\begin{aligned} d(\text{ga-tp}(H_\nu^{-1} \circ H_\eta(\bar{a})/F_\gamma(N_{\gamma,1})), \text{ga-tp}(\bar{a}/N_{\gamma,2})) &= \\ d(\text{ga-tp}(H_\eta(\bar{a})/h_{\rho^\frown 0}(N_{\gamma,1})), \text{ga-tp}(H_\nu(\bar{a})/h_{\rho^\frown 1}(N_{\gamma,2})) &< \varepsilon \end{aligned}$$

(since H_ν is an isometry, $h_{\rho^\frown 0} = h_\rho \circ F_\gamma$, $\rho < \nu$, $\rho^\frown 0 \leq \eta$ and $\rho^\frown 1 \leq \nu$). Since $H_\nu^{-1} \circ H_\eta(\bar{a}) \supset F_\gamma$, then $d(F_\gamma(p \restriction N_{\gamma,1}), p \restriction N_{\gamma,2}) < \varepsilon$, which contradicts the choice of $N_{\gamma,1}$, $N_{\gamma,2}$ and F_γ. $\square_{\text{Claim 1.5.11}}$

We have that $dc(M^*_\zeta) \leq \lambda$ (because $dc(M^*_\zeta) \leq \zeta^* \cdot \zeta = \max\{\mu^+, \zeta\} \cdot \zeta \leq \lambda$). Take $M^* \succ_{\mathcal{K}} M^*_\zeta$ of density character λ; so by claim 1.5.11 we have that $dc(\text{ga-S}(M^*)) \geq 2^\zeta > \lambda$, which contradicts λ-d-stability. $\square_{\text{Lemma 1.5.9}}$

Lemma 1.5.12 (Weak Stationarity over universal models). *For every $\varepsilon > 0$ there exists δ such that for every $N_0 \prec_{\mathcal{K}} N_1 \prec_{\mathcal{K}} N_2$ and every a, b, if N_1 is universal over N_0, $a, b \perp^{T,\delta}_{N_0} N_2$ and $\mathbf{d}(\text{ga-tp}(a/N_1), \text{ga-tp}(b/N_1)) < \delta$, therefore $\mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(b/N_2)) < \varepsilon$.*



Proof. Take $\delta := \delta_\varepsilon/3$ (see definition of tameness, 1.5.2). Let $N^* \prec_{\mathcal{K}} N_0$ be a model of density character $< \zeta^*$ which witnesses $a, b \perp^{T,\delta}_{N_0} N_2$. Let $M^\circ \prec_{\mathcal{K}} N_2$ be a model of density character μ. Let $M^* \prec_{\mathcal{K}} N_2$ be a model of density character $< \zeta^*$ which contains $|N^*| \cup |M^\circ|$. Since N_1 is universal over N_0, it is $< \zeta^*$-universal over N^*. Therefore, there exist a model M' such that $N^* \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} N_1$ and an isomorphism $f : M' \overset{f}{\approx}_{N^*} M^*$. Since N^* witnesses that $a, b \perp^{T,\delta}_{N_0} N_2$ and $N^* \prec_{\mathcal{K}} M' \overset{f}{\approx}_{N^*} M^* \prec_{\mathcal{K}} N_2$, therefore

$$\mathbf{d}(\text{ga-tp}(a/M^*), \text{ga-tp}(f(a)/M^*)) < \delta$$

and

$$\mathbf{d}(\text{ga-tp}(b/M^*), \text{ga-tp}(f(b)/M^*)) < \delta.$$

Also, we have that

$$\begin{aligned}\mathbf{d}(\text{ga-tp}(a/M'), \text{ga-tp}(b/M')) &\leq \mathbf{d}(\text{ga-tp}(a/N_1), \text{ga-tp}(b/N_1)) \\ &< \delta\end{aligned}$$

Therefore:

$$\begin{aligned}\mathbf{d}(\text{ga-tp}(a/M^\circ), \text{ga-tp}(b/M^\circ)) &\leq \mathbf{d}(\text{ga-tp}(a/M^*), \text{ga-tp}(b/M^*)) \\ &\leq \mathbf{d}(\text{ga-tp}(a/M^*), \text{ga-tp}(f(a)/M^*)) \\ &\quad +\mathbf{d}(\text{ga-tp}(f(a)/M^*), \text{ga-tp}(f(b)/M^*)) \\ &\quad +\mathbf{d}(\text{ga-tp}(f(b)/M^*), \text{ga-tp}(b/M^*)) \\ &< 3\delta = \delta_\varepsilon\end{aligned}$$

Since M° is an arbitrary $\mathcal{K}$-submodel of N_2 of density character μ, by μ-$\mathbf{d}$-tameness we have that $\mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(b/N_2)) < \varepsilon$. $\square_{\text{Lemma 1.5.12}}$

CHAPTER 2

Existence and uniqueness of Limit Models in Metric Abstract Elementary Classes

One of the most important classical results in Model theory is the *Morley categoricity transfer theorem* in the setting of countable complete first order theories. The proof of this fact gave us a new useful tool in Model Theory -stability- which has been studied since then.

Shelah stated a conjecture related to this theorem in the setting of (discrete) *AECs*; see [She99]): For any *(discrete) AEC* $\mathcal{K}$ there exists a cardinality μ (which depends on the Löwenheim-Skolem number of $\mathcal{K}$) such that if $\mathcal{K}$ is λ-categorical for some $\lambda \geq \mu$ then $\mathcal{K}$ is κ-categorical for all $\kappa \geq \mu$. Actually, this conjecture has been very hard to prove and we just know some partial answers to this in some special kinds of AECs (e.g.: [She99, SV99, GV06a, HK06, She09a]).

Uniqueness (up to isomorphism) of limit models is a *robust* version of superstability which R. Grossberg and M. VanDieren used in [GV06a] for giving a partial answer -in the setting of *tame* AEC- to the conjecture due to Shelah given above. In fact, such uniqueness plays the same role of saturated models in the original Morley's proof. R. Grossberg, M. VanDieren and A. Villaveces proved in [GVV08] *uniqueness (up to isomorphism) of limit models* assuming that $\mathcal{K}$ does not allow long splitting chains, $\mathcal{K}$ satisfies locality of splitting and $\mathcal{K}$ is κ-Galois-stable (which follow from categoricity). Those superstability-like assumptions hold under categoricity (see [GV06a]).

M. VanDieren in the Bogotá Meeting in Model Theory 2007 talked about the following equivalence stated by S. Shelah (see [Van]): if T is a (countable) complete first order theory, the following are equivalent:

1. T is superstable (i.e.: T is λ-stable for all $\lambda \geq 2^{\aleph_0}$).

2. $\kappa(T) = \aleph_0$ (i.e.: given any infinite ordinal α, there are no a tuple $\overline{a}$ and an increasing chain of sets $\{A_i : i < \alpha\}$ such that $\overline{a} \not\perp_{A_i} A_{i+1}$).

3. Union of an $\prec$-increasing chain of saturated models of T is saturated.

4. Given an $\prec$-increasing and continuous chain $\langle M_i : i < \lambda^+ \rangle$ of models of T of size λ such that $\bigcup_{i<\lambda^+} M_i$ is a saturated model, where $\lambda > |T|^+$ is a regular cardinal, $n_\lambda(T) := \min\{|\{M_\delta/_\approx : \delta \in E\}| : E \subset_{club} \lambda^+\} = 1$ (see [She]).

5. If M_1 and M_2 are Limit Models of T over M_0 with the same cardinality, then $M_1 \approx_{M_0} M_2$ (consequence of [GVV08], because $(Mod(T), \prec)$ is an Abstract Elementary Class).

In (discrete) AECs, *uniqueness of limit models* (up to isomorphism) is a weak notion of superstability. In fact, a suitable version of $\kappa(T) = \aleph_0$ of non-splitting together with some extra assumptions imply such uniqueness.

In our setting (*MAECs*), we also have a version of *Uniqueness of Limit Models*, which is a consequence of some superstability-like assumptions.

In this thesis, we provide a generalization of such uniqueness of limit models, under analogous superstability-like assumptions, but in the setting of *Metric Abstract Elementary Classes*. Although we take some ideas from [GVV08] -in fact, the same sketch of their proof works in our setting-, we have to point out that there are essential differences in our proof with respect to the proof in the discrete case:

1. Existence of limit models is proved in a similar way as in [She09a], but towards getting an actual realization of a type we need to employ a construction in ω many steps, because we cannot find a realization but we can find a realized type which is close enough to such a type (our definition of stability involves a dense subset of the Galois-types space).

2. We adapt most of the definitions given in [GVV08] to our setting. For example, our definition of extension of towers involves $\mathcal{K}$-embeddings, instead of just $\mathcal{K}$- inclusions as in [GVV08]. Also, instead of a notion of a union of an increasing and continuous chain of towers, we have to define a similar notion via directed limits. Although the proof given in [GVV08] does not use directed limit constructions, we have to point out that the proofs of uniqueness of limit models given in [Van06] use directed limits, however M. VanDieren does not define the extension of towers via embeddings.

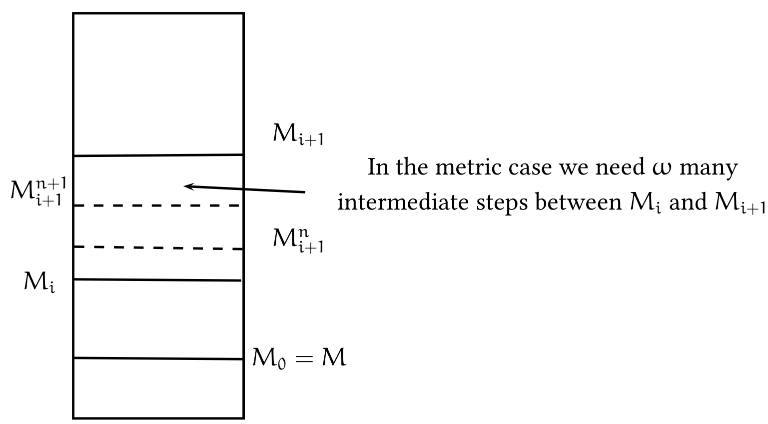
3. We define the notion of *reduced tower* using the notion of *smooth independence* given in chapter 1, instead of just intersections as in [GVV08]. Although we proved that our notion of reduced towers are closed under *directed limits*, are dense and are continuous, the proofs which we provided in this thesis are quite different from the proofs given in [GVV08] because we consider quite different definitions.

In this chapter, we consider the notion of smooth independence defined in chapter 1.

2.1 Existence of limit models in MAEC

In section 1.1 we define the notion of universal model in the setting of Metric Abstract Elementary Classes (see definition 1.3.16). In this section, we provide a proof of the existence of this kind of models. As some proofs in this work, the sketch of the proof is quite similar to the related one in the setting of (discrete) Abstract Elementary Class (see [GV06b]), but some key details are completely different. For example, one of them is the fact that most often we cannot obtain an actual realization of a Galois-type in just one step because of the definition of d-stability; we can just get a realization of a type which is pretty close to the original one. Because of that, we have to use lemma 1.3.22 (which involves a construction with ω-many steps) towards getting an actual realization of a Galois-type.
We now prove the existence of universal extensions in the setting of μ-d-stable Metric Abstract Elementary Classes. We point out that this is an adaptation of the proof of the existence of universal extensions over a given model M in the setting of Abstract Elementary Classes (see [GV06b]). In that proof, under μ-stability, we can consider an increasing and continuous $\mathcal{K}$-chain $\langle M_i : i < \mu \rangle$ such that $M_0 := M$ and where M_{i+1} realizes every Galois-type in ga-S(M_i). So, $\bigcup_{i<\mu} M_i$ is universal over M. But in this setting, we cannot infer directly from μ-**d**-stability that M_{i+1} realizes every type in ga-S(M_i). But we use Lemma 1.3.22 in a suitable way to guarantee that requirement.

Proposition 2.1.1 (Existence of universal extensions). *Let $\mathcal{K}$ be a MAEC μ-**d**-stable with AP. Then for all $M \in \mathcal{K}$ such that $\mathrm{dc}(M) = \mu$ there exists $M^* \in \mathcal{K}$ universal over M such that $\mathrm{dc}(M^*) = \mu$*



Proof. The proof goes almost along the same lines as the proof of existence of universal models in usual AECs (see Claim 2.9 of [GV06b] and Claim 1.15.1 of [She09a]); that is, by trying to capture realizations of types along the construction in a coherent way, and building the universal extension as a union of a chain.
In our metric setting, we need to be careful with the way we realize the types along the construction: although this cannot be done in an immediate way in each successor stage as in [GV06b], lemma 1.3.22 provides the realizations we need between dense subsets of the typespace in ω many steps.

We construct an increasing and continuous $\prec_{\mathcal{K}}$-chain of models $\langle M_i : i < \mu \rangle$ such that $M_0 := M$, $(M^n_{i+1} : n < \omega)$ is a resolution of M_{i+1} where $M^0_{i+1} := M_i$, M^{n+1}_{i+1} realizes a dense subset of ga-$S(M^n_{i+1})$ and $dc(M^n_{i+1}) = \mu$ for every $n < \omega$. This is possible by μ-**d**-stability of $\mathcal{K}$. Take $M^* := \overline{\bigcup_{i<\mu} M_i}$. M^* turns out to be universal over M — by the same argument as in Claim 2.9 of [GV06b]. For the sake of completeness, we write this proof.
Let $N \in \mathcal{K}$ be such that $dc(N) = \mu$ and $M \prec_{\mathcal{K}} N$. We have to construct a $\prec_{\mathcal{K}}$-embedding $f : N \to M^*$ such that $f \upharpoonright M = id_M$.

Let $A := \{a_i : i < \mu\}$ be an enumeration of a dense subset of $N \setminus M$ (i.e. $\overline{A} = N \setminus M$).

Let $\langle N^i_l : l \in \{0,1\}, i < \mu \rangle$ be an increasing and continuous $\prec_{\mathcal{K}}$-chain of models with density character μ and $\langle f_i : i < \mu \rangle$ be an increasing and continuous $\subseteq$-chain of $\prec_{\mathcal{K}}$-embeddings such that:

1. $N^i_0 \prec_{\mathcal{K}} N^i_1$ for all $i < \mu$.
2. $N^0_0 := M$ and $N^0_1 := N$.
3. $a \in N^{i+1}_0$.
4. $f_i : N^i_0 \to M_i$

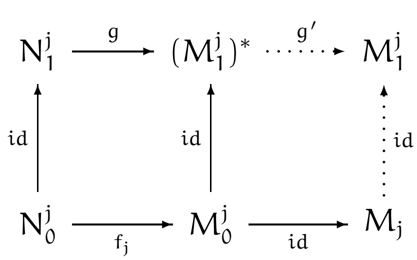
<u>This is enough</u>: Take $g : \overline{\bigcup_{i<\mu} N^i_0} \to M^*$ as the completion of $\bigcup_{i<\mu} f_i$ given by lemma 1.3.20, and $f := g \upharpoonright N$ is the desired $\prec_{\mathcal{K}}$-embedding.

<u>This is possible</u>: If $i < \mu$ is a limit ordinal, take $N^i_l := \overline{\bigcup_{j<i} N^j_l}$ ($l \in \{0,1\}$) and f_i as the completion of $\bigcup_{j<i} f_j$ given by lemma 1.3.20.

If $i := j + 1$ ($j < \mu$), we consider two cases: If $a_i \in N^j_0$, take $N^{j+1}_0 := N^j_0$ and $f_{j+1} := f_j$. Otherwise, as $f_j : N^j_0 \to M_j$ is a $\prec_{\mathcal{K}}$-embedding, we take $M^j_0 := f_j[N^j_0] \prec_{\mathcal{K}} M_j$.

Let $g \supseteq f_j$ and $(M^j_1)^* \in \mathcal{K}$ be such that $g : N^j_1 \xrightarrow{\approx} (M^j_1)^*$. Notice that $(M^j_1)^* \succ_{\mathcal{K}} M^j_0$.

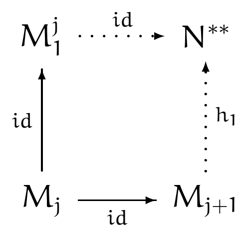
As $\mathcal{K}$ satisfies AP, we have that there exist $M^j_1 \in \mathcal{K}$ and a $\prec_{\mathcal{K}}$-embedding $g' : (M^*_j) \to M^j_1$ such that $M_j \prec_{\mathcal{K}} M^j_1$ and also the following diagram commutes:



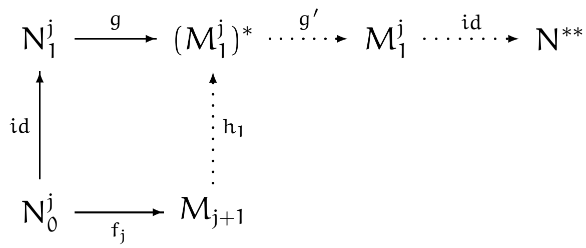
As $a_j \in A \subseteq N = N_1^0 \prec_{\mathcal{K}} N_1^j$, so $a_j \in N_1^j \setminus N_0^j$. Therefore, $g(a_j) \in g[N_1^j] \setminus g[N_0^j] = (M_1^j)^* \setminus M_0^j$. Without loss of generality (by renaming) we can assume that $g'(g(a_j)) \notin M_j$.

Let $p := \text{ga-tp}(g'(g(a_j))/M_j, M_1^j)$. By 1.3.22 and by construction of M_{j+1}, we have that p is realized in M_{j+1}.

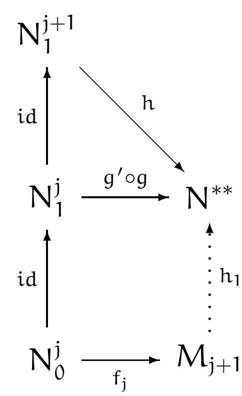
Take $b \models p$ such that $b \in M_{j+1}$, so there exist $N^{**} \in \mathcal{K}$ and a $\prec_{\mathcal{K}}$-embedding $h_1 : M_{j+1} \to N^{**}$ such that $M_1^j \prec_{\mathcal{K}} N^{**}$, $g'(g(a_j)) = h_1(b)$ and the following diagram commutes:



So, we have the following commutative diagram:



By renaming, take $N_1^{j+1} \in \mathcal{K}$ and $h \supseteq g' \circ g$ such that $N_0^j \prec_{\mathcal{K}} N_0^{j+1}$ and $h : N_1^{j+1} \xrightarrow{\cong} N^{**}$.



Take $N_0^{j+1} := h^{-1}[h_1[M_{j+1}]]$ and $f_{j+1} := h_1^{-1} \circ h \restriction N_0^{j+1}$. Note that $N_0^j \prec_{\mathcal{K}} N_0^{j+1}$. $\square_{\text{Prop. 2.1.1}}$

Corollary 2.1.2. *Let $\mathcal{K}$ be a MAEC μ-**d**-stable with AP. Then for all $M \in \mathcal{K}$ such that $\text{dc}(M) = \mu$ there exists $M^* \in \mathcal{K}$ limit over M such that $\text{dc}(M^*) = \mu$.*

Proof. Iterate the construction given in proposition 2.1.1. $\square_{\text{Cor. 2.1.2}}$

2.2 Uniqueness of Limit Models

In this section, we prove *uniqueness -up to isomorphism- of limit models* (i.e., if M_1 is a (μ, θ_1)-limit model over M and M_2 is a (μ, θ_2)-limit model over M and $dc(M_1) = dc(M_2)$, then $M_1 \approx_M M_2$) under suitable superstability-like assumptions. If $cf(\theta_1) = cf(\theta_2)$, then by a standard *back and forth* argument we are done. So, if $cf(\theta_1) \neq cf(\theta_2)$, as in [GVV08], the key idea is to build a (μ, θ)-limit model over M M_θ which is also a (μ, ω)-limit model over M for any ordinal $\theta < \mu^+$, so

$$\begin{aligned} M_1 &\approx_M M_{\theta_1} \text{ (because they are } (\mu, \theta_1)\text{-limits over } M) \\ &\approx_M M_{\theta_2} \text{ (because they are } (\mu, \omega)\text{-limits over } M) \\ &\approx_M M_2 \text{ (because they are } (\mu, \theta_2)\text{-limits over } M) \end{aligned}$$

In order to build that model, as in [GVV08], we define the notion of *smooth tower*, which corresponds to an adaptation of the notion of *tower* given in [GVV08] but in our metric setting. The key idea is to extend (via $\mathcal{K}$-embeddings) a given tower of length of cofinality θ to a special kind of tower (*reduced towers*) which is continuous and to a kind of tower (*relatively full tower*) which satisfies a kind of relative saturation. Iterating this argument ω many times, the idea is to prove that the directed limit of such directed system is a *reduced* (and therefore a continuous) tower where the completion of its union is a (μ, θ)-limit model over M (which is consequence of the *full-relativeness* of the extensions given in the directed system). To be a (μ, ω)-limit model over M is assured defining in a suitable way the notion of extension of towers (see definition 2.2.6). Although this argument has the same sketch of the proof done in [GVV08], we point out that the details in our proof are quite different: e.g., our notion of extension of s-towers involves $\mathcal{K}$-embeddings instead of just $\mathcal{K}$-inclusions as in [GVV08], our notion of *reduced tower* involves the notion of *smooth independence* defined in chapter 1 instead of just intersections as in [GVV08] and since our notion of extension of s-towers involves $\mathcal{K}$-embeddings then we have to consider the notion of *directed limit* of a directed system of towers instead of the union of an increasing chain of towers as in [GVV08].

We split this section in four subsections: The first one sets the general assumptions in our proof and we define the notion of s-tower, which corresponds to an adaptation of the notion of towers in (discrete) AECs to our setting. The second subsection treats the notion of *Reduced s-tower*, and we prove some basic properties of this kind of s-tower (density, closure under directed limits and continuity). The third subsection is about the notion of *relatively full s-towers*, which corresponds to a kind of relative saturation inside a s-tower, which allows us to prove that the final construction in this section is in fact a Limit Model. So, alternating ω many times the density of Reduced s-towers and relatively full s-towers, the directed limit of this construction is a (μ, ω)-Limit Model which is also a (μ, θ)-Limit Model. We provide the details of the proof in section 2.4.

2.2.1 Smooth towers

As we stated above, the key idea in the proof of *uniqueness of limit models* is to build a (μ, θ)-limit model which is also a (μ, ω)-limit model. Smooth tower (shortly, s-tower) corresponds to an adaptation of the notion of tower given in [GVV08], but in our metric setting. In this section, we define the notion of s-tower, extension of s-towers and directed limit of a directed system of s-towers. The key idea is to take a directed limit of a special kind of s-towers (*reduced* s-towers, which will be defined in subsection 2.2.2) which is continuous, where *full-relativeness* (which will be defined in subsection 2.2.3) guarantees that the completion of its union is in fact a (μ, θ)-limit model. To be a (μ, ω)-limit model is guaranteed by definition of extension of s-towers.

Assumption 2.2.1. *Throughout this section, we assume that all our models have density character* $\leq \mu$*, all orderings denoted by* I, I', I_β*, etc. have cardinality* μ *as well, and* $cf(I) = cf(I') = cf(I_\beta) > \omega$*, unless otherwise stated.*

Assumption 2.2.2 (superstability). *For every* a *and every increasing and continuous* $\prec_{\mathcal{K}}$*-chain of models* $\langle M_i : i < \sigma\rangle$ *and* $\mathcal{M}_j$ *a resolution of* M_j *(*$j < \sigma$*):*

1. *(Continuity) If* $p \upharpoonright M_i \downarrow^{\mathcal{M}_0}_{M_0} M_i$ *for all* $i < \sigma$*, then* $p \downarrow^{\mathcal{M}_0}_{M_0} \overline{\bigcup_{i<\sigma} M_i}$.
2. *(Local character) if* $cf(\sigma) > \omega$*, there exists* $j < \sigma$ *such that* $a \downarrow^{\mathcal{M}_j}_{M_j} \bigcup_{i<\sigma} M_i$.
3. *(weak* ε*-simplicity) if* $cf(\sigma) = \omega$*, there exists* $j < \sigma$ *such that* $a \downarrow^{\varepsilon}_{M_j} \overline{\bigcup_{i<\sigma} M_i}$.

Remark 2.2.3. In the continuous setting, we have to split in two different cases the local character assumption, because under superstability, given a finite tuple b and a set B (inside a monster model) there exists a countable set A' of B such that $tp(a/B)$ does not fork over A'. In ω-d-stable homogeneous MAEC, Å. Hirvonen and T. Hyttinen just proved that for every $\epsilon > 0$ there exists a finite subset A of B such that b is ϵ independent from B over A, which takes us to get just a countable subset A of B such that b is independent from B over A

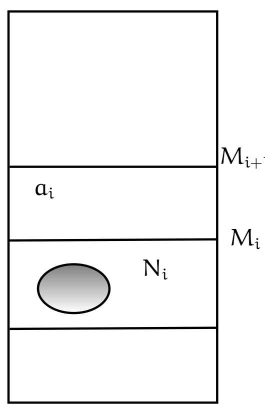
Remark 2.2.4. Notice that assumptions 2.2.2 (2) and (3) imply $a \downarrow^{\mathcal{M}}_{M} M$ for every a and $\mathcal{M}$ a resolution M of M (by monotonicity and definition). Suppose $\mathcal{M} := \{M_i : i < \sigma\}$.

1. If $cf(\sigma) = \omega$, given $\varepsilon > 0$, by ε-simplicity there exists $i_\varepsilon < \sigma$ such that $a \downarrow^{\varepsilon}_{M_{i_\varepsilon}} M$. By definition, this means $a \downarrow^{\mathcal{M}}_{M} M$.
2. If $cf(\sigma) > \omega$, then by assumption 2.2.2 (2) (local character) there exists $i < \sigma$ such that $a \downarrow^{\mathcal{M}_i}_{M_i} M$. Since $M_i \prec_{\mathcal{K}} M$, by monotonicity of s-independence we have $a \downarrow^{\mathcal{M}}_{M} M$.

Anyway, under assumption 2.2.2 (2) and (3), we have $a \downarrow^{\mathcal{M}}_{M} M$. We call assumption 2.2.2 (3) *weak* ε*-simplicity*, because in (discrete) first order theories, simplicity (non existence of a formula with the tree property, see [TZ09, Wag00]) implies $tp(a/A)$ does not fork over A. Also, T. Hyttinen and M. Kesälä used $a \downarrow_C C$ (where $\downarrow$ is a well-behaved independence notion defined in the context of finitary abstract elementary classes) for every tuple a and every set C as a version of simplicity (see [HK06]). In first order, simplicity corresponds to a more general setting than stability, where non-forking still has good properties. In the setting of finitary AECs, under simplicity a suitable notion of independence $\downarrow$ also has good properties, but Hirvonen and Hyttinen are interested in proving a categoricity transfer theorem in their setting.

As in (discrete) AECs, a (μ, θ)-limit model is witnessed by a $\prec_{\mathcal{K}}$-increasing and continuous chain of models $(M_i : I < \theta)$ in $\mathcal{K}$ such that for any $i \in I$ we have that M_{i+1} is universal over M_i. By μ-d-stability, we can find a model (of density character μ) $N_i \prec_{\mathcal{K}} M_i$ such that $a_I \downarrow_{N_i}^{\mathcal{N}_i} M_i$ (and moreover, such N_i can be chosen such that M_i is a (μ, σ)-limit model over N_i, by assumption 2.2.2). So, we can define the notion of s-tower in the following way:

Definition 2.2.5 (smooth towers). Let I be a well-ordering, $\mathfrak{M} := (M_i : i \in I)$ be an $\prec_{\mathcal{K}}$-increasing chain, $\overline{a} := (a_i : i \in I)$, $\mathfrak{N} := (N_i : i < \sigma)$ be a sequence of models in $\mathcal{K}$, $\mathcal{M} := (\mathcal{M}_j : j \in I)$ be a sequence of resolutions $\mathcal{M}_j$ of M_j $(j \in I)$ and $\mathcal{N} := (\mathcal{N}_j : j \in I)$ be a sequence of resolutions $\mathcal{N}_j$ of N_j $(j \in I)$. We say that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is a *smooth tower* (shortly, *s-tower*) iff for every $i \in I$ we have that M_i is a (μ, σ)-limit model over N_i for some $\sigma < \mu^+$, $a_i \in M_{i+1} \setminus M_i$ and $a_i \downarrow_{N_i}^{\mathcal{N}_i} M_i$.



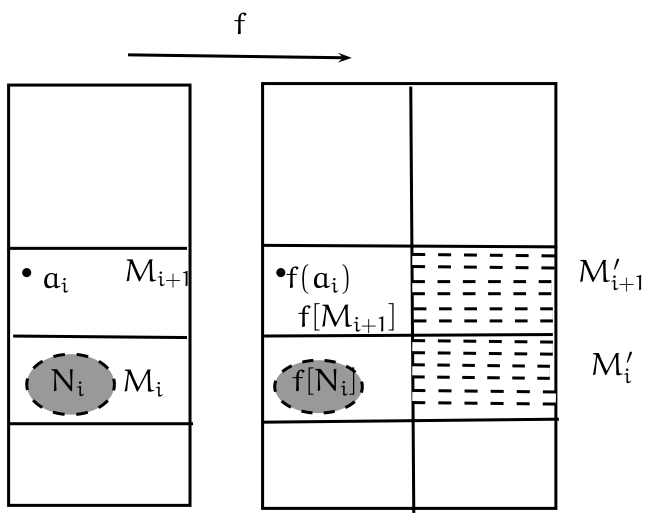
Roughly speaking, an s-tower is composed by a $\prec_{\mathcal{K}}$-increasing (not necessarily continuous) chain of models $\mathfrak{M} := (M_i : i \in I)$ and a collection of $\mathcal{K}$-submodels $\mathfrak{N} := (N_i : i \in I)$ such that each M_i is a (μ, σ)-limit model over N_i (for some $\sigma < \mu^+$) which codify a smooth independence of the elements a_i taken in the s-tower (i.e., $a_i \downarrow_{N_i}^{\mathcal{N}_i} M_i$).

The following definition is an adaptation of the notion of extension of towers given in [GVV08, SV99]. In the definition given in those papers, they just consider $\mathcal{K}$-extensions. In this work, we consider extensions via $\mathcal{K}$-embeddings instead of just inclusions, because in our adaptation of the proof of continuity of our notion of reduced towers, directed systems appear in crucial parts of the proof. Following the same philosophy as in (discrete) AECs, we require that every model M_i' in the extension indexed by an element in the set of indexed of the extended s-tower is a universal model over an isomorphic copy of the respective model M_i in the extended s-tower.

So, when we take the directed limit of a directed system of ω many s-towers, the corresponding directed limit is a (μ, ω)-limit model

Definition 2.2.6 (Extension of s-towers). Let $I \leq I'$ be well-orderings, $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}_{\mu,I}$ and $(\mathfrak{M}', \overline{a}', \mathfrak{N}', M', N') \in \mathcal{K}_{\mu,I'}$. We say that $(\mathfrak{M}', \overline{a}', \mathfrak{N}', M', N')$ extends $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ (which we denote by $(\mathfrak{M}', \overline{a}', \mathfrak{N}', M', N') > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$) iff there exists a $\mathcal{K}$-embedding $f : \overline{\bigcup_{i \in I} M_i} \to \overline{\bigcup_{j \in I'} M'_j}$ such that for every $i \in I$:

1. M'_i is a proper universal model over $f[M_i]$
2. $f[\mathcal{M}_i] \subset \mathcal{M}'_i$.
3. $f(a_i) = a'_i$
4. $f[N_i] = N'_i$
5. $f[\mathcal{N}_i] = \mathcal{N}'_i$



Definition 2.2.7 (Weak directed limit of an $<$-increasing chain of towers). Let $\langle I_\alpha : \alpha < \beta \rangle$ be an $\subset$-increasing chain of well-orderings and suppose that $I := \bigcup_{\alpha<\beta} I_\alpha$ is a well-ordering. Let $\langle (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha : \alpha < \beta \rangle$ be an $\leq$-increasing chain of towers (witnessed by a system of mappings $(f_{\alpha,\gamma} : \alpha \leq \gamma < \beta)$), where $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha \in \mathcal{K}_{\mu,I_\alpha}$. We define the *weak directed limit* of this $\leq$-increasing chain as follows:

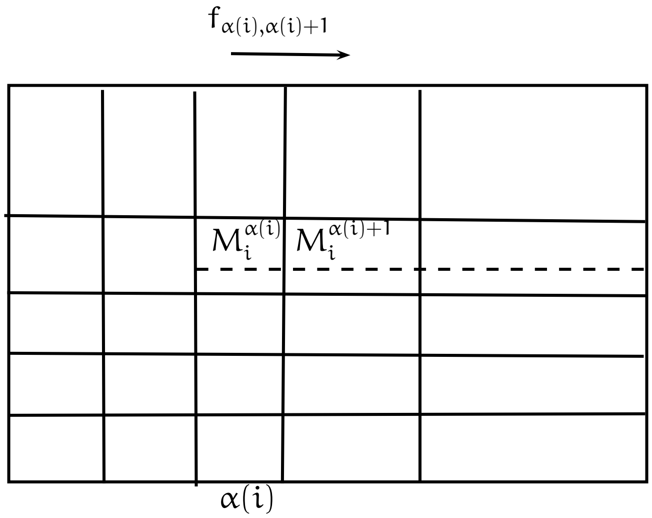
1. For every $i \in I$, let $\alpha(i) := \min\{\alpha < \beta : i \in I_\alpha\}$.
2. For $i \in I$, define $M := \lim_{\rightarrow} \langle \overline{\bigcup_{i \in I_\alpha} M_i^\alpha}; f_{\alpha,\gamma} : \alpha \leq \gamma < \beta \rangle$, with canonical embeddings denoted by $f_{\alpha,\beta}$. Define $M_i := \overline{\bigcup_{\alpha(i) \leq \alpha < \beta} f_{\alpha,\beta}[M_i^\alpha]}$ (this definition makes sense because if

$\alpha(i) \leq \alpha < \gamma < \beta$, then $f_{\alpha,\beta}[M_i^\alpha] = (f_{\gamma,\beta} \circ f_{\alpha,\gamma})[M_i^\alpha] \prec_{\mathcal{K}} f_{\gamma,\beta}[M_i^\gamma])$, $a_i := f_{\alpha(i),\beta}(a_i^{\alpha(i)})$ and $N_i := f_{\alpha(i),\beta}[N_i^{\alpha(i)}]$. Define $\mathfrak{M} := (M_i : i \in I)$, $\overline{a} := (a_i : i \in I)$, $\mathfrak{N} := (N_i : i \in I)$, $\mathcal{M} := (\mathcal{M}_i : i \in I)$ where each $\mathcal{M}_i$ is defined as the respective concatenations of the resolutions $f_{\alpha,\beta}[\mathcal{M}_i^\alpha]$ for $\alpha(i) \leq \alpha < \beta$ and $\mathcal{N} := (f_{\alpha(i),\beta}[\mathcal{N}_i^{\alpha(i)}] : i \in I)$.

Fact 2.2.8. $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *is in fact an s-tower.*

Proof. Notice that M_i is universal over N_i (since $f_{\alpha,\beta}[M_i^\alpha]$ is universal over N_i), $a_i \mathop{\smile\hspace{-0.9em}\perp}_{N_i}^{\mathcal{N}_i} f_{\alpha,\beta}[M_i^\alpha]$ (by invariance), so by superstability (assumption 2.2.2) $a_i \mathop{\smile\hspace{-0.9em}\perp}_{N_i}^{\mathcal{N}_i} M_i$. Therefore $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ defined as above is in fact a tower which $\leq$-extends every tower $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha$ □

The s-tower $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ defined as above is called *the weak directed limit* of $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha$ $\alpha < \beta$

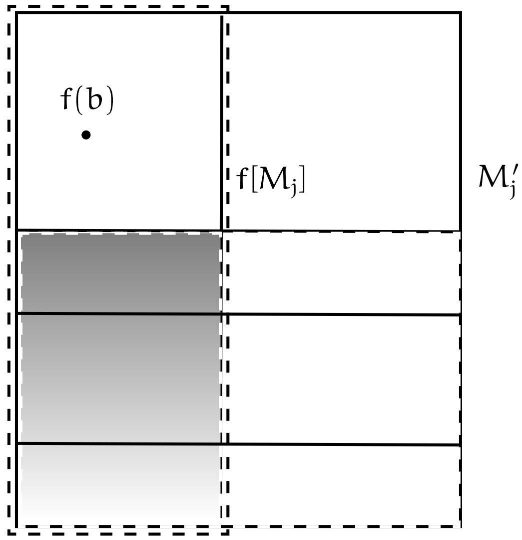


2.2.2 Reduced s-Towers

The following is an adaptation of the notion of *reduced tower* given in [GVV08], but using the new notion of extension of towers introduced in this thesis. In (discrete) AECs, reduced towers are such towers that their extensions do not add more information in terms of intersections (see [GVV08]). Our definition follows the same idea, but not adding more information in terms of *smooth independence.*

As in (discrete) AECs, our notion of *reduced s-towers* corresponds to a special kind of s-towers which satisfy the following properties: any s-tower can be extended to a reduced s-tower (density of reduced s-towers) and if $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is reduced then $\mathfrak{M}$ is continuous. Continuity is very important in the proof of *uniqueness of limit models* because we require that the final tower in this construction satisfies that the completion of its union is in fact a (μ, θ)-limit model (in particular, we require that the last model in the final s-tower is in fact the completion of the union of every previous model in the final s-tower).

Definition 2.2.9. We say that $(\mathfrak{M}, \overline{a}, \mathfrak{N})$ is a *reduced s-tower* iff for every extension $(\mathfrak{M}', \overline{a}', \mathfrak{N}', \mathcal{M}', \mathcal{N}') > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ (witnessed by a $\mathcal{K}$-embedding f) and for every $j \in I$ we have that $f[\overline{\bigcup_{i \in I} M_i}] \downarrow_{f[M_j]}^{f[\mathcal{M}_j]} M'_j$ (i.e.: for every $b \in \overline{\bigcup_{i\in I} M_i}$, $f(b) \downarrow_{f[M_j]}^{f[\mathcal{M}_j]} M'_j$).



Roughly speaking, *reduced s-towers* are s-towers whose their extensions do not add more information in terms of *smooth independence.*

The following facts are adaptations of the analogous properties proved in [GVV08] which reduced towers satisfy.

Proposition 2.2.10 (density of reduced s-towers). *Every s-tower* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *has an extension which is a reduced s-tower.*

Proof. Suppose not. So, we can construct an $<$-increasing and continuous chain $\langle (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha : \alpha < \mu^+ \rangle$ such that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\alpha+1}$ witnesses that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha$ is not reduced (via $f_{\alpha,\alpha+1} : \overline{\bigcup_{i\in I} M_i^\alpha} \to \overline{\bigcup_{i\in I} M_i^{\alpha+1}}$). For $\alpha < \delta \leq \mu^+$ (δ a limit ordinal), define $f_{\alpha,\delta}$ as the canonical $\mathcal{K}$-embedding given by the corresponding directed limit and for $\alpha < \beta \leq \mu^+$ (β a successor ordinal), take $f_{\alpha,\beta}$ as the corresponding natural compositions.

Notice that if $\alpha < \beta < \mu^+$, then $f_{\alpha,\mu^+}[M_i^\alpha] \prec_{\mathcal{K}} f_{\beta,\mu^+}[M_i^\beta]$ (since $f_{\alpha,\beta}[M_i^\alpha] \prec_{\mathcal{K}} M_i^\beta$, so applying f_{β,μ^+} we have $f_{\alpha,\mu^+}[M_i^\alpha] = (f_{\beta,\mu^+} \circ f_{\alpha,\beta})[M_i^\alpha] \prec_{\mathcal{K}} f_{\beta,\mu^+}[M_i^\beta]$).

For a fixed $l \in I$, by assumption 2.2.2 (superstability-like assumptions), for every c we have that there exists a minimal $\beta < \mu^+$ such that $c \downarrow_{f_{\beta,\mu^+}[M_l^\beta]}^{f_{\beta,\mu^+}[\mathcal{M}_l^\beta]} \overline{\bigcup_{\alpha<\mu^+} f_{\alpha,\mu^+}[M_l^\alpha]}$ (which we denote by α_c^l). Define $\alpha_c := \sup\{\alpha_c^l : l \in I\} < \mu^+$ (because $|I| \leq \mu$). Notice that for every $i \in I$, $f_{\alpha_c^i,\mu^+}[M_i^{\alpha_c^i}] \prec_{\mathcal{K}} f_{\alpha_c,\mu^+}[M_i^{\alpha_c}] \prec_{\mathcal{K}} \overline{\bigcup_{\alpha<\mu^+} f_{\alpha,\mu^+}[M_i^\alpha]}$, so by monotonicity we have that $c \mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\alpha_c,\mu^+}[\mathcal{M}_l^{\alpha_c}]}_{f_{\alpha_c,\mu^+}[M_l^{\alpha_c}]} \overline{\bigcup_{\alpha<\mu^+} f_{\alpha,\mu^+}[M_l^{\alpha}]}$ (since $f_{\alpha_c^i,\mu^+}[M_i^{\alpha_c^i}] \prec_{\mathcal{K}} f_{\alpha_c,\mu^+}[M_i^{\alpha_c}]$).

Take any $\gamma_0 < \mu^+$. We have that $\bigcup_{i\in I} f_{\gamma_0,\mu^+}[M_i^{\gamma_0}]$ has density character μ, so we can take $B_{\gamma_0} \subseteq \bigcup_{i\in I} f_{\gamma_0,\mu^+}[M_i^{\gamma_0}]$ of cardinality μ such that $\overline{B_{\gamma_0}} = \bigcup_{i\in I} f_{\gamma_0,\mu^+}[M_i^{\gamma_0}]$. Defining a mapping $c \mapsto \alpha_c < \mu^+$ for every $c \in B_{\gamma_0}$, we have that there exists $\gamma_0' < \mu^+$ such that $\alpha_c < \gamma_0'$ for every $c \in B_{\gamma_0}$. Take $\gamma_1 := \max\{\gamma_0, \gamma_0'\} + 1$.

Suppose that we have defined $\gamma_n < \mu^+$ for a fixed $n < \omega$. Take $B_{\gamma_n} \subseteq \bigcup_{i\in I} f_{\gamma_n,\mu^+}[M_i^{\gamma_n}]$ of cardinality μ such that $\overline{B_{\gamma_n}} = \bigcup_{i\in I} f_{\gamma_n,\mu^+}[M_i^{\gamma_n}]$. Defining a mapping $c \mapsto \alpha_c < \mu^+$ for every $c \in B_{\gamma_n}$, we have that there exists $\gamma_n' < \mu^+$ such that $\alpha_c < \gamma_n'$ for every $c \in B_{\gamma_n}$. Take $\gamma_{n+1} := \max\{\gamma_n, \gamma_n'\} + 1$.

We have that $\gamma := \sup\{\gamma_n : n < \omega\} < \mu^+$.

Take $b \in \overline{\bigcup_{i\in I} f_{\gamma,\gamma+1}[M_i^{\gamma}]}$ which witnesses that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\gamma}$ is not reduced: i.e.: there exists $k \in I$ such that $b \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\gamma,\gamma+1}[\mathcal{M}_k^{\gamma}]}_{f_{\gamma,\gamma+1}[M_k^{\gamma}]} M_k^{\gamma+1}$. We know that there exists (b_n) in $\bigcup_{i\in I} f_{\gamma,\gamma+1}[M_i^{\gamma}]$ such that $(b_n) \to b$, so by continuity of smooth independence there exists $N < \omega$ such that $b_N \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\gamma,\gamma+1}[\mathcal{M}_k^{\gamma}]}_{f_{\gamma,\gamma+1}[M_k^{\gamma}]} M_k^{\gamma+1}$. By definition, there exists some $j \in I$ such that $b_N \in f_{\gamma,\gamma+1}[M_j^{\gamma}] = \overline{\bigcup_{\alpha<\gamma} f_{\alpha,\gamma+1}[M_j^{\alpha}]}$. Notice that $j > k$.

Since $b_N \in f_{\gamma,\gamma+1}[M_j^{\gamma}] = \overline{\bigcup_{\alpha<\gamma} f_{\alpha,\gamma+1}[M_j^{\alpha}]}$, there exists a sequence (c_n) in $\bigcup_{\alpha<\gamma} f_{\alpha,\gamma+1}[M_j^{\alpha}]$ such that $(c_n) \to b_N$. Since $b_N \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\gamma,\gamma+1}[\mathcal{M}_k^{\gamma}]}_{f_{\gamma,\gamma+1}[M_k^{\gamma}]} M_k^{\gamma+1}$, by continuity of $\mathop{\smile\!\!\!\!\!\!|}$ there exists $M < \omega$ such that $c_M \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\gamma,\gamma+1}[\mathcal{M}_k^{\gamma}]}_{f_{\gamma,\gamma+1}[M_k^{\gamma}]} M_k^{\gamma+1}$.

Notice that there exists $\alpha_M < \gamma$ such that $c_M \in f_{\alpha_M,\gamma+1}[M_j^{\alpha_M}]$. By definition of γ, there exists $L < \omega$ such that $\alpha_M < \gamma_L$, therefore $c_M \in f_{\gamma_L,\gamma+1}[M_j^{\gamma_L}]$, so $f_{\gamma+1,\mu^+}(c_M) \in f_{\gamma_L,\mu^+}[M_j^{\gamma_L}] \subseteq \bigcup_{i\in I} f_{\gamma_L,\mu^+}[M_i^{\gamma_L}]$. Take a sequence $(d_n) \in B_{\gamma_L}$ such that $(d_n) \to f_{\gamma+1,\mu^+}(c_M)$. By invariance, we may say $f_{\gamma+1,\mu^+}(c_M) \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\gamma,\mu^+}[\mathcal{M}_k^{\gamma}]}_{f_{\gamma,\mu^+}[M_k^{\gamma}]} f_{\gamma+1,\mu^+}[M_k^{\gamma+1}]$; by continuity of $\mathop{\smile\!\!\!\!\!\!|}$ there exists $K < \omega$ such that $d_K \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\gamma,\mu^+}[\mathcal{M}_k^{\gamma}]}_{f_{\gamma,\mu^+}[M_k^{\gamma}]} f_{\gamma+1,\mu^+}[M_k^{\gamma+1}]$. Let $c := d_K$, so $\alpha_c < \gamma_{L+1} < \gamma$. Since $c \mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\alpha_c,\mu^+}[\mathcal{M}_k^{\alpha_c}]}_{f_{\alpha_c,\mu^+}[M_k^{\alpha_c}]} \overline{\bigcup_{\alpha<\mu^+} f_{\alpha,\mu^+}[M_k^{\alpha}]}$, then by monotonicity we have $c \mathop{\smile\!\!\!\!\!\!|}\nolimits^{f_{\gamma,\mu^+}[\mathcal{M}_k^{\gamma}]}_{f_{\gamma,\mu^+}[M_k^{\gamma}]} f_{\gamma+1,\mu^+}[M_k^{\gamma+1}]$ (contradiction). $\square_{\text{Prop. 2.2.10}}$

Proposition 2.2.11 (weak directed limits of reduced s-towers).
If $\langle((\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\gamma} : \gamma < \beta\rangle$ is an $<$-increasing and continuous chain of reduced s-towers (witnessed by $f_{\gamma,\alpha}$ for $\gamma \leq \alpha < \beta$; where $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\gamma} \in \mathcal{K}_{\mu,I_\gamma}$ and $I_\alpha \subseteq I_\gamma$ if $\alpha < \gamma < \beta$), the directed limit of this sequence is a reduced s-tower indexed by $I_\beta := \bigcup_{\gamma<\beta} I_\gamma$.

Proof. Suppose not. Let $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\beta}$ be an s-tower and $f : \overline{\bigcup_{i\in I} M_i^{\beta}} \to \overline{\bigcup_{i\in I} M_i}$ witnessing that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\beta}$ is not a reduced s-tower. Let $b \in \overline{\bigcup_{i\in I_\beta} f[M_i^{\beta}]} =$

$\bigcup_{i\in I_\beta} f[M_i^\beta]$ (because $cf(I_\beta) > \omega$) and $i \in I_\beta$ be such that $b \not\downarrow_{f[M_i^\beta]}^{f[\mathcal{M}_i^\beta]} M_i$. Also, there exists $j \in I_\beta$ such that $b \in f[M_j^\beta]$. Notice that $j > i$.

Since $b \in f[M_j^\beta] := f[\overline{\bigcup_{\alpha(j)\leq\alpha<\beta} f_{\alpha,\beta}[M_j^\alpha]}]$ (remember $\alpha(j) := \min\{\gamma < \beta : j \in I_\gamma\}$), there exists a sequence (b_n) in $\bigcup_{\alpha(j)\leq\alpha<\beta}(f \circ f_{\alpha,\beta})[M_j^\alpha]$ such that $(b_n) \to b$. By continuity of s-independence, there exists $N < \omega$ such that $b_N \not\downarrow_{f[M_i^\beta]}^{f[\mathcal{M}_i^\beta]} M_i$.

Let $\alpha(i) := \min\{\gamma < \beta : i \in I_\gamma\}$ and γ be such that $\max\{\alpha(i), \alpha(j)\} \leq \gamma < \beta$ and $b_N \in (f \circ f_{\gamma,\beta})[M_j^\gamma] \subseteq \bigcup_{i\in I_\gamma}(f \circ f_{\gamma,\beta})[M_i^\gamma]$. Since $b_N \not\downarrow_{f[M_i^\beta]}^{f[\mathcal{M}_i^\beta]} M_i$, by monotonicity we have that $b_N \not\downarrow_{(f\circ f_{\gamma,\beta})[M_i^\gamma]}^{(f\circ f_{\gamma,\beta})[\mathcal{M}_i^\gamma]} M_i$. This contradicts the fact that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\gamma$ is a reduced s-tower.

$\square_{\text{Prop. 2.2.11}}$

Proposition 2.2.12 (Truncation of reduced towers). *Let* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}_{\mu,I}$ *be a reduced tower and* $I' \subset I$ *an initial segment of* I. *Then* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \upharpoonright I'$ *is a reduced tower.*

Proof. Suppose not, so there exists an initial segment $I_0 \subset I$ such that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \upharpoonright I_0$ is not a reduced tower. Hence there exist an extension $(\mathfrak{M}', \overline{a}', \mathfrak{N}', M', N') \geq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \upharpoonright I_0$ (witnessed by f_0), $b \in \overline{\bigcup_{i\in I_0} M_i}$ and $i \in I_0$ such that $b \not\downarrow_{f_0[M_i]}^{f_0[\mathcal{M}_i]} M_i'$.

Let $\delta := \min[I \setminus I_0]$. Let $\delta^- := \delta$ if δ is a limit element and $\delta^- := \delta - 1$ if δ is a successor element. Notice that $f_0 : \overline{\bigcup_{i<\delta} M_i} \to \overline{\bigcup_{i<\delta} M_i'}$. Let $\overline{f}_0 \in Aut(\mathbb{M})$ be such that $\overline{f}_0 \supset f_0$. Let M_{δ^-}' be a universal model over $f_0[M_{\delta^-}]$ which contains $\overline{\bigcup_{i<\delta} M_i'}$ and $N_{\delta^-}' := f_0[N_{\delta^-}]$. By definition of tower, $a_{\delta^-} \downarrow_{N_{\delta^-}}^{\mathcal{N}_{\delta^-}} M_{\delta^-}$. By invariance, $\overline{f}_0(a_{\delta^-}) \downarrow_{\overline{f}_0[N_{\delta^-}]}^{\overline{f}_0[\mathcal{N}_{\delta^-}]} \overline{f}_0[M_{\delta^-}]$. By extension property (proposition 1.4.9), there exists $g_0 \in Aut(\mathbb{M}/\overline{f}_0[M_{\delta^-}])$ such that $(g_0 \circ \overline{f}_0)(a_{\delta^-}) \downarrow_{\overline{f}_0[N_{\delta^-}]}^{\overline{f}_0[\mathcal{N}_{\delta^-}]} M_{\delta^-}'$. Define $\overline{f}_1 := (g_0 \circ \overline{f}_0)$ and $f_1 := \overline{f}_1 \upharpoonright M_{\delta^-} = \overline{f}_0 \upharpoonright M_{\delta^-} : M_{\delta^-} \to M_{\delta^-}'$. Notice that $f_1 \supseteq f_0$. Define M_{δ^-+1}' as a universal model over $\overline{f}_1[M_{\delta^-+1}]$ which contains M_{δ^-}'. Notice that $\overline{f}_1(a_{\delta^-}) \in \overline{f}_1[M_{\delta^-+1}] \subset M_{\delta^-+1}'$ and $\overline{f}_1(a_{\delta^-}) \notin M_{\delta^-}'$ (by proposition 1.4.12) Proceed in a similar way for obtaining f_i for every $i \in I \setminus I_0$, getting and increasing $\subset$-chain of embeddings and define $g := \overline{\bigcup_{i\in I} f_i}$. Notice that the tower and g defined in this way witness that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is not a reduced tower (contradiction). $\square_{\text{Prop. 2.2.12}}$

Proposition 2.2.13 (reduced towers are continuous). *If* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *is a reduced tower, then it is continuous.*

Proof. Let $\mathfrak{T}_0 := ((\langle M_i : i \in I\rangle, \langle a_i : i \in I\rangle, \langle N_i : i \in I\rangle, \langle \mathcal{M}_i : i \in I\rangle, \langle \mathcal{N}_i : i \in I\rangle)$ be a reduced s-tower which is not continuous. Let δ be the minimum limit ordinal such that continuity fails in $\mathfrak{T}_0$ at δ; i.e.: there exists $b \in M_\delta \setminus \overline{\bigcup_{i<\delta} M_i}$. Without loss of generality, by truncation of reduced towers -Proposition 2.2.12-, we may assume that $I := \delta + 1$. By density of reduced towers, let $\mathfrak{T}_1 := (\langle M_i^1 : i \leq \delta\rangle, \langle f_{0,1}(a_i) : i \leq \delta\rangle, \langle f_{0,1}[N_i] : i \leq \delta\rangle, \langle \mathcal{M}_i^1 : i \leq$

$\delta\rangle, \langle f_{0,1}[\mathcal{N}_i] : i \leq \delta\rangle)$ be a reduced tower which extends $\mathfrak{T}_0$ via $f_{0,1}$. For $\beta = \alpha + 1 < \delta$, extend $\mathfrak{T}_\alpha := (\langle M_i^\alpha : i \leq \delta\rangle, \langle f_{0,\alpha}(a_i) : i \leq \delta\rangle, \langle f_{0,\alpha}[N_i] : i \leq \delta\rangle, \langle \mathcal{M}_i^1 : i \leq \delta\rangle, \langle f_{0,\alpha}[\mathcal{N}_i] : i \leq \delta\rangle)$ to a reduced tower $(\langle M_i^{\alpha+1} : i \leq \delta\rangle, \langle f_{\alpha,\alpha+1} \circ f_{0,\alpha}(a_i) : i \leq \delta\rangle, \langle f_{\alpha,\alpha+1} \circ f_{0,\alpha}[N_i] : i \leq \delta\rangle, \langle \mathcal{M}_i^1 : i \leq \delta\rangle, \langle f_{\alpha,\alpha+1} \circ f_{0,\alpha}[\mathcal{N}_i] : i \leq \delta\rangle)$ via $f_{\alpha,\alpha+1}$ $(\alpha < \delta)$. Define $f_{\alpha,\gamma}$ $(\alpha \leq \gamma < \delta)$ inductively as follows: for $\gamma = \alpha$, define $f_{\alpha,\alpha} := id$; given $f_{\alpha,\gamma}$, define $f_{\alpha,\gamma+1} := f_{\gamma,\gamma+1} \circ f_{\alpha,\gamma}$. For $\beta < \delta$ limit, define $\mathfrak{T}_\beta$ as the *weak* directed limit of the towers $\langle \mathfrak{T}_\alpha, f_{\alpha,\gamma} : \alpha \leq \gamma < \beta\rangle$, with $f_{\alpha,\beta} : M_\delta^\alpha \to M_\delta^\beta$ the corresponding canonical embedding.
Let $\mathfrak{T}_\delta := (\langle M_i^\delta : i \leq \delta\rangle, \langle f_{0,\delta}(a_i) : i \leq \delta\rangle, \langle f_{0,\delta}[N_i] : i \leq \delta\rangle, \langle \mathcal{M}_i^1 : i \leq \delta\rangle, \langle f_{0,\delta}[\mathcal{N}_i] : i \leq \delta\rangle)$ be the *weak* directed limit of the towers $(\mathfrak{T}_\alpha; f_{\alpha,\beta} : \alpha \leq \beta < \delta)$ with canonical embeddings $f_{\alpha,\delta}$ $(\alpha < \delta)$.

Remark 2.2.14 ($f_{0,\alpha}(b) \notin M_\alpha^\alpha$ for every $\alpha < \delta$). If at some stage $\alpha < \delta$ $f_{0,\alpha}(b)$ appeared inside M_α^α, we have that $f_{0,\alpha}(b) \not\perp_{f_{0,\alpha}[M_\alpha]} M_\alpha^\alpha$ because $f_{0,\alpha}(b) \in M_\alpha^\alpha \setminus f_{0,\alpha}[M_\alpha]$ (by antirreflexivity of smooth independence, Proposition 1.4.12). $\mathfrak{T}_\alpha$ would then witness that $\mathfrak{T}_0$ is not a reduced tower (contradiction). Therefore, there is no any $\alpha < \delta$ such that $f_{0,\alpha}(b)$ appears inside M_α^α

By assumption 2.2.2 (2) and (3), we have that either for any fixed $\varepsilon > 0$ there exists $\zeta < \delta$ such that $f_{0,\delta}(b) \perp^{\varepsilon}_{f_{\zeta,\delta}[M_\zeta^\zeta]} \overline{\bigcup_{\alpha<\delta} f_{\alpha,\delta}[M_\alpha^\alpha]}$ (if $cf(\delta) = \omega$, apply (3)) or there exists $\zeta < \delta$ such that $f_{0,\delta}(b) \perp^{f_{\zeta,\delta}[\mathcal{M}_\zeta^\zeta]}_{f_{\zeta,\delta}[M_\zeta^\zeta]} \overline{\bigcup_{\alpha<\delta} f_{\alpha,\delta}[M_\alpha^\alpha]}$ (if $cf(\delta) > \omega$, apply (2)).

Without loss of generality, for the sake of simplicity, we may assume $\zeta = 0$.

CASE $i = 0$: Define $N'_0 := f_{1,\delta}[M_0^1]$. Notice that $f_{1,\delta}[M_0^1] \prec_{\mathcal{K}} f_{1,\delta}[M_1^1]$, since $M_0^1 \prec_{\mathcal{K}} M_1^1$. Let $h_0 := id \restriction f_{0,\delta}[M_0^0]$. Notice that $h_0[f_{0,\delta}[M_0^0]] = f_{0,\delta}[M_0^0] \prec_{\mathcal{K}} f_{1,\delta}[M_0^1]$. Thus $h_0 : f_{0,\delta}[M_0^0] \to f_{1,\delta}[M_0^1] \prec_{\mathcal{K}} N'_0$.

CASE $i = 1$: Let $h_1 := id \restriction f_{1,\delta}[M_1^1] : f_{1,\delta}[M_1^1] \to f_{2,\delta}[M_1^2]$. Notice that $h_0 \subset h_1$. Since $f_{0,1}(a_0) \perp^{f_{0,1}[\mathcal{N}_0]}_{f_{0,1}[N_0]} M_0^1$ (since $\mathfrak{T}_0 \leq \mathfrak{T}_1$), by invariance -applying $f_{1,\delta}$- we have

$$f_{0,\delta}(a_0) \perp^{f_{0,\delta}[\mathcal{N}_0]}_{f_{0,\delta}[N_0]} f_{1,\delta}[M_0^1] \tag{2-1}$$

Let N'_1 be a universal model over $\overline{\bigcup_{n<\delta} f_{n,\delta}[M_n^n]}$ containing $f_{0,\delta}(b)$. Notice that $N'_0 := f_{1,\delta}[M_0^1] \prec_{\mathcal{K}} f_{1,\delta}[M_1^1] \prec_{\mathcal{K}} N'_1$. and $h_1 : f_{1,\delta}[M_1^1] \to f_{2,\delta}[M_1^2] \prec_{\mathcal{K}} f_{2,\delta}[M_2^2] \prec_{\mathcal{K}} N'_1$. Let $l_{0,1} := id \restriction N'_0 : N'_0 \to N'_1$.

Notice that $f_{0,1}(a_0) \in M_1^1$ (since $\mathfrak{T}_1$ is a tower), then $f_{0,\delta}(a_0) = f_{1,\delta} \circ f_{0,1}(a_0) \in f_{1,\delta}[M_1^1]$. Therefore, (2-1) means

$$h_1 \circ f_{0,\delta}(a_0) \perp^{h_0 \circ f_{0,\delta}[\mathcal{N}_0]}_{h_0 \circ f_{0,\delta}[N_0]} l_{0,1}[N'_0] \tag{2-2}$$

CASE $i = 2$: Since $f_{0,1}(a_1) \downarrow^{f_{0,1}[\mathcal{N}_1]}_{f_{0,1}[N_1]} M^1_1$ (definition of $\leq$ and since $\mathfrak{T}_1$ is a tower), by invariance and the commutative property of directed limits, applying $f_{1,\delta}$ we have

$$f_{0,\delta}(a_1) \downarrow^{f_{0,\delta}[\mathcal{N}_1]}_{f_{0,\delta}[N_1]} f_{1,\delta}[M^1_1] \tag{2-3}$$

Since $f_{1,\delta}[M^1_1]$ is universal over $f_{0,\delta}[N_1]$ and $f_{1,\delta}[M^1_1] \prec_{\mathcal{K}} f_{2,\delta}[M^2_1] \prec_{\mathcal{K}} N'_1$, by extension - Proposition 1.4.9- there exists $g_1 \in Aut(\mathbb{M}/f_{1,\delta}[M^1_1])$ such that

$$g_1 \circ f_{0,\delta}(a_1) \downarrow^{f_{0,\delta}[\mathcal{N}_1]}_{f_{0,\delta}[N_1]} N'_1 \tag{2-4}$$

Notice that $f_{3,\delta}[M^3_2]$ is universal over $f_{2,\delta}[M^2_2]$ (because $\mathfrak{T}_2 \leq \mathfrak{T}_3$), in particular it is universal over $f_{1,\delta}[M^1_1] \prec_{\mathcal{K}} f_{2,\delta}[M^2_2]$. Since $g_1 \circ f_{2,\delta}[M^2_2] \succ_K f_{1,\delta}[M^1_1]$, there exists $f_1 : g_1 \circ f_{2,\delta}[M^2_2] \to f_{3,\delta}[M^3_2]$ fixing $f_{1,\delta}[M^1_1]$ pointwise. Let

$$h_2 := f_1 \circ g_1 \restriction f_{2,\delta}[M^2_2] : f_{2,\delta}[M^2_2] \to f_{3,\delta}[M^3_2]$$

Notice that $h_1 \subset h_2$, since both f_1 and g_1 fix $f_{1,\delta}[M^1_1]$ pointwise and $h_1 := id \restriction f_{1,\delta}[M^1_1]$.

Let $\bar{f}_1$ be an automorphism of $\mathbb{M}$ extending f_1. By invariance, applying $\bar{f}_1$ to (4) we get

$$f_1 \circ g_1 \circ f_{0,\delta}(a_1) \downarrow^{f_{0,\delta}[\mathcal{N}_1]}_{f_{0,\delta}[N_1]} \bar{f}_1[N'_1]. \tag{2-5}$$

Let N'_2 be a universal model over $f_{3,\delta}[M^3_2]$ containing $\bar{f}_1[N'_1]$ and $l_{1,2} := \bar{f}_1 \restriction N'_1 : N'_1 \to N'_2$. Notice that $h_2 : f_{2,\delta}[M^2_2] \to f_{3,\delta}[M^3_2] \prec_{\mathcal{K}} N'_2$ and $f_{0,1}(a_1) \in f_{2,\delta}[M^2_2]$.

Therefore (2-5) means

$$h_2 \circ f_{0,\delta}(a_1) \downarrow^{h_1 \circ f_{0,\delta}[\mathcal{N}_1]}_{h_1 \circ f_{0,\delta}[N_1]} l_{1,2}[N'_1]. \tag{2-6}$$

CASE $i = 3$: Since $f_{0,2}(a_2) \downarrow^{f_{0,2}[\mathcal{N}_2]}_{f_{0,2}[N_2]} M^2_2$ (definition of $\leq$ and since $\mathfrak{T}_2$ is a tower), by invariance and the commutative property of directed limits, applying $f_{2,\delta}$ we have

$$f_{0,\delta}(a_2) \downarrow^{f_{0,\delta}[\mathcal{N}_2]}_{f_{0,\delta}[N_2]} f_{2,\delta}[M^2_2] \tag{2-7}$$

Let $\bar{h}_2$ be an automorphism of $\mathbb{M}$ extending h_2. By invariance,

$$\bar{h}_2 \circ f_{0,\delta}(a_2) \downarrow^{h_2 \circ f_{0,\delta}[N_2]}_{h_2 \circ f_{0,\delta}[N_2]} h_2 \circ f_{2,\delta}[M_2^2] \tag{2-8}$$

Since $h_2 \circ f_{2,\delta}[M_2^2]$ is universal over $h_2 \circ f_{0,\delta}[N_2]$ and $h_2 \circ f_{2,\delta}[M_2^2] \prec_{\mathcal{K}} N_2'$, by extension (Proposition 1.4.9) there exists

$$g_2 \in Aut(\mathbb{M}/h_2 \circ f_{2,\delta}[M_2^2])$$

such that

$$g_2 \circ \bar{h}_2 \circ f_{0,\delta}(a_2) \downarrow^{h_2 \circ f_{0,\delta}[N_2]}_{h_2 \circ f_{0,\delta}[N_2]} N_2' \tag{2-9}$$

Notice that $f_{4,\delta}[M_3^4]$ is universal over $f_{3,\delta}[M_3^3]$ (because $\mathfrak{T}_3 \leq \mathfrak{T}_4$), in particular it is universal over $h_2[f_{2,\delta}[M_2^2]] \prec_{\mathcal{K}} f_{3,\delta}[M_2^3] \prec_{\mathcal{K}} f_{3,\delta}[M_3^3]$. Since $g_2 \circ \bar{h}_2 \circ f_{3,\delta}[M_3^3] \succ_K h_2[f_{2,\delta}[M_2^2]]$, there exists

$$f_2 : g_2 \circ \bar{h}_2 \circ f_{3,\delta}[M_3^3] \to f_{4,\delta}[M_3^4]$$

fixing $h_2[f_{2,\delta}[M_2^2]]$ pointwise.

Let $h_3 := f_2 \circ g_2 \circ \bar{h}_2 \upharpoonright f_{3,\delta}[M_3^3] : f_{3,\delta}[M_3^3] \to f_{4,\delta}[M_3^4]$.

Notice that $h_2 \subset h_3$, since both f_2 and g_2 fix $h_2[f_{2,\delta}[M_2^2]]$ pointwise.

Let $\bar{f}_2$ be an automorphism of $\mathbb{M}$ extending f_2. Applying invariance to (2-9), we have

$$f_2 \circ g_2 \circ \bar{h}_2 \circ f_{0,\delta}(a_2) \downarrow^{h_2 \circ f_{0,\delta}[N_2]}_{h_2 \circ f_{0,\delta}[N_2]} \bar{f}_2[N_2'] \tag{2-10}$$

Let N_3' be a universal model over $f_{4,\delta}[M_3^4]$ containing $\bar{f}_2[N_2']$. Notice that $h_3 : f_{3,\delta}[M_3^3] \to f_{4,\delta}[M_3^4] \prec_{\mathcal{K}} N_3'$. Let $l_{2,3} := \bar{f}_2 \upharpoonright N_2' : N_2' \to N_3'$. Notice that $f_{0,\delta}(a_2) \in f_{3,\delta}[M_3^3]$.

Therefore, (2-10) means

$$h_3 \circ f_{0,\delta}(a_2) \downarrow^{h_2 \circ f_{0,\delta}[N_2]}_{h_2 \circ f_{0,\delta}[N_2]} l_{2,3}[N_2'] \tag{2-11}$$

Continue in a similar way for $n < \omega$.

CASE $i = \omega$: Notice that $i < j < \omega$ implies

$$f_{i,\delta}[M^i_i] \prec_{\mathcal{K}} f_{j,\delta}[M^j_j]:$$

$f_{i,\delta}[M^i_i] = f_{j,\delta}[f_{i,j}[M^i_i]] \prec_{\mathcal{K}} f_{j,\delta}[M^j_j]$ (since $f_{i,\delta} = f_{j,\delta} \circ f_{i,j}$).
Ê
Let $\hat{N}_\omega$ be the directed limit of $\langle N'_n, l_{n,m} : n < \omega, n < m < \omega \rangle$, with canonical embeddings $l_{n,\omega} : N'_n \to \hat{N}_\omega$.
Let $k_n := l_{n,\omega} \circ h_n : f_{n,\delta}[M^n_n] \to l_{n,\omega}[f_{n+1,\delta}[M^{n+1}_n]]] \prec_{\mathcal{K}} l_{n,\omega}[N'_n] \prec_{\mathcal{K}} \hat{N}_\omega$. Notice that $k_n \subset k_{n+1}$: Let $a \in f_{n,\delta}[M^n_n]$, then

$$\begin{aligned}
k_n(a) &= l_{n,\omega} \circ h_n(a) \\
&= (l_{n+1,\omega} \circ l_{n,n+1}) \circ h'_n(a) \text{ (by commutative} \\
&\quad \text{properties of canonical embeddings of directed systems)} \\
&= l_{n+1,\omega} \circ (l_{n,n+1} \circ h_n)(a) \\
&= l_{n+1,\omega} \circ (\bar{f}_n \circ h_n)(a) \ (l_{n,n+1} := \bar{f}_n \restriction N'_n \text{, by definition)} \\
&= l_{n+1,\omega} \circ (f_n \circ g_n \circ h_n)(a) \ (g_n \text{ fixes } h_n \circ f_{n,\delta}[M^n_n] \text{ pointwise and } \bar{f}_n \supset f_n) \\
&= l_{n+1,\omega} \circ h_{n+1}(a) \ (h_{n+1} = f_n \circ g_n \circ \bar{h}_n \restriction f_{n+1,\delta}[M^{n+1}_{n+1}]) \\
&= k_{n+1}(a) \text{ (by definition of } k_{n+1}\text{)}.
\end{aligned}$$

Let $h_\omega := \overline{\bigcup_{n<\omega} k_n} : \overline{\bigcup_{n<\omega} f_{n,\delta}[M^n_n]} \to \overline{\bigcup_{n<\omega} l_{n,\omega}[f_{n+1,\delta}[M^{n+1}_n]]} \prec_{\mathcal{K}} \hat{N}_\omega$.

Notice that

$$\begin{aligned}
l_{n,n+1} \circ h_n \restriction f_{n,\delta}[M^n_n] &= \bar{f}_n \circ h_n \restriction f_{n,\delta}[M^n_n] \\
&\quad (l_{n,n+1} := \bar{f}_n \restriction N'_n \text{ and } h_n[f_{n,\delta}[M^n_n]] \prec_{\mathcal{K}} N'_n) \\
&= f_n \circ g_n \circ h_n \restriction f_{n,\delta}[M^n_n] \\
&\quad \text{(since } g_n \text{ fixes } h_n \circ f_{n,\delta}[M^n_n] \text{ pointwise and} \\
&\quad \bar{f}_n \supset f_n) \\
&= h_{n+1} \restriction f_{n,\delta}[M^n_n] \\
&= h_n \restriction f_{n,\delta}[M^n_n] \text{ (since } h_{n+1} \supset h_n)
\end{aligned}$$

Therefore, we conclude that $l_{n,n+1}$ fixes $h_n[f_{n,\delta}[M^n_n]]$ pointwise. Without loss of generality, doing a pull back, we may assume that $l_{n,\omega}$ fixes $h_n[f_{n,\delta}[M^n_n]]$ pointwise.

1. Case 1: $\omega < \delta$: Let N'_ω be a universal model over $h_\omega[f_{\omega,\delta}[M^\omega_\omega]] = h_\omega[\overline{\bigcup_{n<\omega} f_{n,\delta}[M^n_n]}]$ containing $\hat{N}_\omega$.

   Notice that $f_{\omega,\delta}[M^\omega_\omega] = \overline{\bigcup_{n<\omega} f_{\omega,\delta}[M^\omega_n]} = \overline{\bigcup_{n<\omega} f_{n,\delta}[M^n_n]}$ (since $\omega < \delta$ and δ is the minimum ordinal such that there exists a reduced tower with a failure of continuity at level δ).

Notice that

$$\begin{aligned}
h_\omega[\overline{\bigcup_{n<\omega} f_{n,\delta}[M_n^n]}] &= \overline{\bigcup_{n<\omega} k_n[f_{n,\delta}[M_n^n]]} \\
&= \overline{\bigcup_{n<\omega} l_{n,\omega} \circ h_n[f_{n,\delta}[M_n^n]]} \\
&= \overline{\bigcup_{n<\omega} h_n[f_{n,\delta}[M_n^n]]} \\
&\quad (l_{n,\omega} \text{ is assumed to fix } f_{n,\delta}[M_n^n] \text{ pointwise}) \\
&\prec_{\mathcal{K}} \overline{\bigcup_{n<\omega} f_{n+1,\delta}[M_n^{n+1}]} \\
&= \overline{\bigcup_{n<\omega} f_{n,\delta}[M_n^n]}
\end{aligned}$$

Therefore $h_\omega : f_{\omega,\delta}[M_\omega^\omega] \to f_{\omega,\delta}[M_\omega^\omega] \prec_{\mathcal{K}} f_{\omega+1,\delta}[M_\omega^\omega + 1]$.

Since $f_{0,\omega}(a_\omega) \downarrow_{f_{0,\omega}[N_\omega]}^{f_{0,\omega}[\mathcal{N}_\omega]} M_\omega^\omega$ (since $\mathfrak{T}_0 \leq \mathfrak{T}_\omega$), by invariance and the commutative property of directed limits, applying $f_{\omega,\delta}$ we have

$$f_{0,\delta}(a_\omega) \downarrow_{f_{0,\delta}[N_\omega]}^{f_{0,\delta}[\mathcal{N}_\omega]} f_{\omega,\delta}[M_\omega^\omega] \tag{2-12}$$

Let $\bar{h}_\omega \in Aut(\mathbb{M})$ be extending h_ω. By invariance, we have that

$$\bar{h}_\omega \circ f_{0,\delta}(a_\omega) \downarrow_{h_\omega \circ f_{0,\delta}[N_\omega]}^{h_\omega \circ f_{0,\delta}[\mathcal{N}_\omega]} h_\omega \circ f_{\omega,\delta}[M_\omega^\omega]. \tag{2-13}$$

Since $h_\omega \circ f_{\omega,\delta}[M_\omega^\omega]$ is universal over $h_\omega \circ f_{0,\delta}[N_\omega]$ and $h_\omega \circ f_{\omega,\delta}[M_\omega^\omega] \prec_{\mathcal{K}} N'_\omega$, by extension -Proposition 1.4.9- there exists $g_\omega \in Aut(\mathbb{M}/h'_\omega \circ f_{\omega,\delta}[M_\omega^\omega])$ such that

$$g_\omega \circ \bar{h}_\omega \circ f_{0,\delta}(a_\omega) \mathop{\smile\!\!\!\!\!\!|}\;_{h_\omega \circ f_{0,\delta}[N_\omega]}^{h_\omega \circ f_{0,\delta}[\mathcal{N}_\omega]} N'_\omega \tag{2-14}$$

Notice that $f_{\omega+2,\delta}[M^{\omega+2}_{\omega+1}]$ is universal over $f_{\omega+1,\delta}[M^{\omega+1}_{\omega+1}]$, in particular it is universal over $h_\omega[f_{\omega,\delta}[M^\omega_\omega]] \prec_{\mathcal{K}} f_{\omega,\delta}[M^\omega_\omega] \prec_{\mathcal{K}} f_{\omega+1,\delta}[M^{\omega+1}_{\omega+1}]$.

Let $\bar{h}_\omega \in Aut(\mathbb{M})$ extending h_ω. Since $g_\omega \circ \bar{h}_\omega[f_{\omega+1,\delta}[M^{\omega+1}_{\omega+1}]] \succ_K h_\omega[f_{\omega,\delta}[M^\omega_\omega]]$, there exists $f_\omega : g_\omega \circ \bar{h}_\omega[f_{\omega+1,\delta}[M^{\omega+1}_{\omega+1}]] \to \bar{h}_\omega[f_{\omega+2}[M^{\omega+2}_{\omega+1}]]$ fixing $h_\omega[f_{\omega,\delta}[M^\omega_\omega]]$ pointwise.

Let $\bar{f}_\omega \in Aut(\mathbb{M})$ extending f_ω, then by invariance applied to (2-14) we get

$$f_\omega \circ g_\omega \circ \bar{h}_\omega \circ f_{0,\delta}(a_\omega) \mathop{\smile\!\!\!\!\!\!|}\;_{h_\omega \circ f_{0,\delta}[N_\omega]}^{h_\omega \circ f_{0,\delta}[\mathcal{N}_\omega]} \bar{f}_\omega[N'_\omega] \tag{2-15}$$

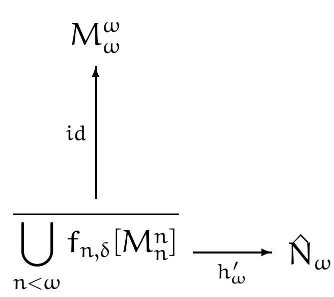
Let $h_{\omega+1} := f_\omega \circ g_\omega \circ \bar{h}_\omega \upharpoonright f_{\omega+1,\delta}[M^{\omega+1}_{\omega+1}] : f_{\omega+1,\delta}[M^{\omega+1}_{\omega+1}] \to h'_\omega[f_{\omega+2,\delta}[M^{\omega+2}_{\omega+1}]]$. Notice that $h_\omega \subset h_{\omega+1}$ because both f_ω and g_ω fix $h_\omega[f_{\omega,\delta}[M^\omega_\omega]]$ pointwise.

Let $N'_{\omega+1}$ be a universal model over $f_{\omega+1,\delta}[M^{\omega+1}_\omega] \succ_K h_\omega[f_{\omega,\delta}[M^\omega_\omega]]$ containing $\bar{f}_\omega[N'_\omega]$ and $l_{\omega,\omega+1} := \bar{f}_\omega \upharpoonright N'_\omega : N'_\omega \to N'_{\omega+1}$.

Since $f_{0,\delta}(a_\omega) \in f_{\omega+1,\delta}[M^{\omega+1}_{\omega+1}]$, notice that (2-15) can be written as follows:

$$h_{\omega+1} \circ f_{0,\delta}(a_\omega) \mathop{\smile\!\!\!\!\!\!|}\;_{h_\omega \circ f_{0,\delta}[N_\omega]}^{h_\omega \circ f_{0,\delta}[\mathcal{N}_\omega]} l_{\omega,\omega+1}[N'_\omega] \tag{2-16}$$

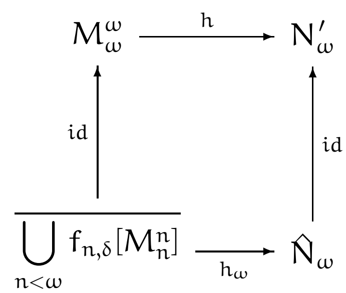
2. Case 2: $\omega = \delta$: We have the following diagram:



We want to find $\bar{h} \in Aut(\mathbb{M})$ such that $\bar{h}(f_{0,\delta}(b)) = l_{1,\omega}(f_{0,\delta}(b))$, extending h_ω. We just know that for a fixed $\varepsilon > 0$ there exists $\zeta < \delta$ -recall that *without loss of generality*, we may assume $\zeta = 0$- such that $f_{0,\delta}(b) \downfree^{\varepsilon}_{f_{0,\delta}[M^0_0]} \overline{\bigcup_{i<\delta} f_{i,\delta}[M^i_i]}$. We conjecture that under the stronger condition $f_{0,\delta}(b) \downfree^{f_{0,\delta}[\mathcal{M}^0_0]}_{f_{0,\delta}[M^0_0]} \overline{\bigcup_{i<\delta} f_{i,\delta}[M^i_i]}$ we could find such automorphism in a similar way as it was proved in [GVV08] that there exists an automorphism $\bar{h}$ of the monster model such that $\bar{h}(b) = b$, extending some mapping h with domain $\bigcup_{i<\delta} M^i_i$. Suppose that we were succeeded finding such automorphism $\bar{h}$.

Let $h := \bar{h} \upharpoonright M^\omega_\omega$ and N'_ω be a universal model over $h[M^\omega_\omega]$ containing $\hat{N}_\omega$.

Since $h \supset h_\omega$, notice that the following diagram commutes:



Define $j := h \circ f_{0,\delta} : M^0_\omega \to N'_\omega$. Notice that by equations (2-2), (2-6) and (2-11), we can assure that

$$h_{i+1}(f_{0,\delta}(a_i)) \downfree^{h_i \circ f_{0,\delta}[\mathcal{N}_i]}_{h_i \circ f_{0,\delta}[N_i]} l_{i,i+1}[N'_i] \tag{2-17}$$

Since $h_{i+1} \supset h_i$, we have that

$$h_{i+1}(f_{0,\delta}(a_i)) \downfree^{h_{i+1} \circ f_{0,\delta}[\mathcal{N}_i]}_{h_{i+1} \circ f_{0,\delta}[N_i]} l_{i,i+1}[N'_i]$$

By invariance, applying $l_{i+1,\omega}$ we get

$$l_{i+1,\omega} \circ h_{i+1}(f_{0,\delta}(a_i)) \downfree^{l_{i+1,\omega} \circ h_{i+1} \circ f_{0,\delta}[\mathcal{N}_i]}_{l_{i+1,\omega} \circ h_{i+1} \circ f_{0,\delta}[N_i]} l_{i,\omega}[N'_i]$$

Since $l_{i+1} \circ h_{i+1} =: k_{i+1} \subset h_\omega \subset h$, we get

$$h(f_{0,\delta}(a_i)) \downfree^{h \circ f_{0,\delta}[\mathcal{N}_i]}_{h \circ f_{0,\delta}[N_i]} l_{i,\omega}[N'_i]$$

i.e.,

$$j(a_i) \mathop{\smile\!\!\!\!\!\mid}\nolimits_{j[N_i]}^{j[\mathcal{N}_i]} l_{i,\omega}[N_i'] \tag{2-18}$$

By antirreflexivity -Prop. 1.4.12-, we have that $j(a) \notin l_{i,\omega}[N_i']$ because $j(a_i) \notin j[N_i]$ and $j(a_i) \mathop{\smile\!\!\!\!\!\mid}\nolimits_{j[N_i]}^{j[\mathcal{N}_i]} l_{i,\omega}[N_i']$.

Therefore, the tower based on $\langle l_{i,\omega}[N_i'] : i < \omega\rangle$, $\langle j[N_i] : i < \omega\rangle$ and $\langle j(a_i) : i < \omega\rangle$ extends $\mathfrak{T}_0$ via j.

Notice that $j(b) = h \circ f_{0,\delta}(b) = l_{1,\omega} \circ f_{0,\delta}(b) \in l_{1,\omega}[N_1'] \setminus h[f_{0,\delta}[M_1^0]] = j[M_1^0]$, then by antirreflexivity -Proposition 1.4.12- we have that $j(b) \not\mathop{\smile\!\!\!\!\!\mid}\nolimits_{j[M_1^0]}^{j[\mathcal{M}_1^0]} l_{1,\omega}[N_1']$, contradicting the reducibility of $\mathfrak{T}_0$.

For $i < \delta$ a successor ordinal, proceed as in the case $i = 3$. If $i < \delta$ is a limit ordinal, proceed as in the case $\omega < \delta$. And finally, for $i = \delta$ proceed as in the case $\omega = \delta$. $\square_{\text{Prop. 2.2.13}}$

2.2.3 Full-relativeness of s-towers

As in (discrete) AECs, *relatively full s-towers* are a kind of s-towers which are relatively saturated. *Reduced s-towers* guarantee that the completion of the union of our directed system is continuous, *relatively full s-towers* guarantee that such completion satisfies that every model indexed by a successor ordinal is in fact universal over its predecessor model (i.e., such model is in fact a (μ, θ)-limit model).

Before defining *relatively full s-towers*, we have to define a notion of strong type as in [GVV08, SV99].

Definition 2.2.15 (strong type). Let M be a σ-limit model

1. $$\mathfrak{St}(M) := \left\{ (p, N) : \begin{array}{l} N \prec_{\mathcal{K}} M \\ N \text{ is a } (\mu,\theta)\text{-limit model} \\ M \text{ is universal over } N \\ p \in \text{ga-S}(M) \text{ is non-algebraic} \\ \text{and } p \mathop{\smile\!\!\!\!\!\mid}\nolimits_{N}^{\mathcal{N}} M \\ \text{for some resolution } \mathcal{N} \text{ of } N. \end{array} \right\}$$
2. Two strong types $(p_l, N_l) \in \mathfrak{St}(M_l)$ ($l \in \{1, 2\}$) are *parallel* (which we denote by $(p_1, N_1) \parallel (p_2, N_2)$) iff for every $M' \succ_{\mathcal{K}} M_1, M_2$ with density character μ, there exists $q \in \text{ga-S}(M')$ which extends both p_1 and p_2 and $q \mathop{\smile\!\!\!\!\!\mid}\nolimits_{N_l}^{\mathcal{N}_l} M'$ ($l \in \{1, 2\}$) (where $\mathcal{N}_l$ is the resolution of N_l which satisfies $p_i \mathop{\smile\!\!\!\!\!\mid}\nolimits_{N_l}^{\mathcal{N}_l} M_l$).

Assumption 2.2.16. *Through this subsection, assume that* I *is a well order which has a cofinal sequence* $(i_\alpha : \alpha < \theta)$*, where* $\mathrm{cf}(\theta) > \omega$.

Definition 2.2.17 (Metric s-Towers). An s-tower $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is called a *metric s-tower* if the resolution witnessing that M_i is a (μ, σ)-limit model over N_i is spread-out. A spread-out resolution $\mathcal{M}$ of M is a resolution where for every γ, $M^{\gamma+1}$ is an ω_1-limit over M^γ.

Definition 2.2.18 (relatively full s-towers). Let $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ be an s-tower indexed by I. Let $(M_i^\gamma : \gamma < \sigma)$ be a sequence which witnesses that M_i is a (μ, σ)-limit model. We say that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is a relatively full s-tower with respect to $(M_i^\gamma)_{i\in I, \gamma<\sigma}$ iff for every $i_\alpha \leq i < i_{\alpha+1}$ and $(p, M_i^\gamma) \in \mathfrak{St}(M_i)$ there exists $i \leq j < i_{\alpha+1}$ such that $(p, M_i^\gamma) \parallel (\text{ga-tp}(a_j/M_j), N_j)$.

Proposition 2.2.19. *Suppose that for every* $\alpha < \theta$ *there are* $\mu\cdot\omega$ *many elements between* i_α *and* $i_{\alpha+1}$*. Let* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *be a relatively full s-tower with respect to* $(M_i^\gamma)_{i\in I,\gamma<\sigma}$*. Then* $M := \overline{\bigcup_{i\in I} M_i}$ *is a limit model over* M_{i_0}.

Proof. It is enough to prove that $M_{i_{\alpha+1}}$ is universal over M_{i_α}. Let $p := \text{ga-tp}(a/M_{i_\alpha}) \in \text{ga-S}(M_{i_\alpha})$ and $\varepsilon > 0$. So, by assumption 2.2.2 there exists $\gamma := \gamma_\varepsilon < \sigma$ such that $a \downarrow^{\varepsilon}_{M_{i_0}^{\gamma_\varepsilon}} M_{i_0}$.

By construction, $M_{i_\alpha}^{\gamma+1}$ is a (μ, ω_1)-limit model over $M_{i_\alpha}^\gamma$. Let $(M_i^* : i < \omega_1)$ be a resolution which witnesses that.

Consider $q := p \restriction M_{i_\alpha}^{\gamma+1}$, so by assumption 2.2.2 there exists $i < \omega_1$ such that $q \downarrow^{\mathcal{M}_i^*}_{M_i^*} M_{i_\alpha}^{\gamma+1}$. By extension over universal models (proposition 1.4.9) (notice that $M_{i_\alpha}^{\gamma+1}$ is universal over M_i^*), there exists $q^* \in \text{ga-S}(M_{i_\alpha})$ an extension of q such that $q^* \downarrow^{\mathcal{M}_i^*}_{M_i^*} M_{i_\alpha}$. So, $(q^*, M_i^*) \in \mathfrak{St}(M_{i_\alpha})$. By relatively fullness of $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$, there exists $i_\alpha \leq j_1 < i_{\alpha+1}$ such that $(q^*, M_i^*) \parallel (\text{ga-tp}(a_j/M_j), N_j)$. Therefore, $q^* = \text{ga-tp}(a_j/M_{i_\alpha})$ and so q^* is realized in M_{j_1}.

By monotonicity of non-ε-splitting, we have that p does not ε-split over M_i^* (since p does not ε-split over $M_{i_\gamma}^\gamma$ and $M_{i_\alpha}^\gamma \prec_{\mathcal{K}} M_i^*$); i.e. $p \downarrow^{\varepsilon}_{M_i^*} M_{i_\alpha}$. Since $q^* \downarrow^{\mathcal{M}_i^*}_{M_i^*} M_{i_\alpha}$, then $q^* \downarrow^{\varepsilon}_{M_i^*} M_{i_\alpha}$ (by monotonicity of non-ε-splitting).

Also, since $q = p \restriction M_{i_\alpha}^{\gamma+1}$ and $q^* \supset q$, then $q^* \restriction M_{i_\alpha}^{\gamma+1} = p \restriction M_{i_\alpha}^{\gamma+1}$. Notice that $M_{i_\alpha}^{\gamma+1}$ is universal over M_i^*.

Since $p, q^* \downarrow^{\varepsilon}_{M_i^*} M_{i_\alpha}$, by a weak version of the stationarity (Lemma 1.4.8), we have that $\mathbf{d}(p, q^*) < 2\varepsilon$. Therefore, M_{j_1} realizes a dense subset of $\text{ga-S}(M_{i_\alpha})$.

Doing a similar argument, we can construct an increasing sequence $(j_n : n < \omega)$ in I (where $j_0 := i_\alpha$) such that $i_\alpha \leq j_n < i_{\alpha+1}$, where $M_{j_{n+1}}$ realizes a dense subset of $\text{ga-S}(M_{j_n})$.

Therefore, by lemma 1.3.22 we have that $M^* := \overline{\bigcup_{n<\omega} M_{j_n}} \prec_{\mathcal{K}} M_{i_{\alpha+1}}$ realizes every type over $M_{j_0} = M_{i_\alpha}$, so $M_{i_{\alpha+1}}$ does.

$\square_{\text{Prop. 2.2.19}}$

The following fact is proved in a similar way as the discrete case (see [GVV08]). For the sake of completeness, we give a proof of this result.

Proposition 2.2.20. *If* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}^*_{\mu,I}$, *there exists* $(\mathfrak{M}', a, \mathfrak{N}, M', N) > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *in* $\mathcal{K}^*_{\mu,I}$ *such that for every limit* $i \in I$, M'_i *is a* (μ, μ)*-limit over* $\overline{\bigcup_{j<i} M_j}$

Proof. First, we construct by induction on $i \in I$ a model $M^+_i \succ_{\mathcal{K}} M_i$ and a directed system $(f_{i,j} : i < j \in I)$ of $\prec_{\mathcal{K}}$-embeddings (as in the discrete AEC case, one may prove that the "union axioms" for metric AEC also hold for directed systems) such that $f_{i,j} : M^+_i \to M^+_j$ and $f_{i,j} \restriction M_i = id_{M_i}$.

Suppose $(M^+_k : k \le i)$ and $(f_{k,l} : k < l \le i)$ are constructed. We give the construction of M^+_{i+1} and $f_{i,i+1}$. The construction of $f_{j,i+1}$ $(j < i)$ are given by definition of directed system. Let M^*_{i+1} be a limit model over M^+_i and M_{i+1}. Since $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathcal{N}_{i+1}}_{N_{i+1}} M_{i+1}$ and M_{i+1} is universal over N_{i+1} (by definition of s-tower), by the extension property ([VZ10b, 2.7]) and invariance of smooth independence there exists $f \in Aut(\mathbb{M}/M_{i+1})$ such that $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathcal{N}_{i+1}}_{N_{i+1}} f[M^*_{i+1}]$. Define $M^+_{i+1} := f[M^*_{i+1}]$ and $f_{i,i+1} := f \restriction M^+_i$.

For limit $i \in I$, first take the directed limit of $(M^+_k : k \le i)$ and $(f_{k,l} : k < l \le i)$ and then consider M^+_i a limit model over this directed limit and (μ, μ)-limit over $\overline{\bigcup_{j<i} f_{j,i}[M^+_j]}$.

Fix $j \in I$. Let $f_{j,\sup(I)}$ and $M'_{j,\sup(I)}$ be the respective directed limit of this directed system. Without loss of generality, we may assume that $f_{j,\sup(I)} \restriction M_j = id_{M_j}$ (just doing a pull back). Define $M'_j := f_{j,\sup(I)}[M^+_j]$.

Notice that the s-tower $(\mathfrak{M}', \overline{a}, \mathfrak{N}, M, N)$ defined in this way satisfies the requirements of the proposition and that $(\mathfrak{M}', \overline{a}, \mathfrak{N}, M, N)$ is not necessarily continuous. $\square_{\text{Prop. 2.2.20}}$

Lemma 2.2.21 (weak relatively full). *Given* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}^*_{\mu,I_n}$, *there exists* $(\mathfrak{M}', a, \mathfrak{N}, M', N) > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *in* $\mathcal{K}^*_{\mu,I_{n+1}}$ *such that for every* $(p, N) \in \mathfrak{St}(M_i)$ *(where* $i \in I_n$ *and* $i_\alpha \le i < i_{\alpha+1}$*) there exists* $i \le j < i_{\alpha+1}$ *such that* $(\text{ga-tp}(a_j/M'_j), N_j) \parallel (p, N)$.

Proof. Let $M'_{i_{\alpha+1}}$ be a (μ, μ)-limit model over $\overline{\bigcup_{j<i_{\alpha+1}, j \in I_n} M_j}$ (by proposition 2.2.20). Let $\langle M'_l : l \in I_{n+i}, i_\alpha + \mu \cdot n < l < \alpha + 1 \rangle$ be an enumeration of a resolution which witnesses that $M'_{i_{\alpha+1}}$ is (μ, μ)-limit over $\overline{\bigcup_{j<i_{\alpha+1}, j \in I_n} M_j}$.

Let $\mathfrak{S} := \{(p,N)^l_\alpha : i_\alpha + \mu\cdot n < l < i_{\alpha+1}\}$ be an enumeration of a dense subset of $\bigcup\{\mathfrak{S}t(M_i) : i \in I_n, i_\alpha \leq i < i_{\alpha+1}\}$ (by μ-stability). Therefore, given $(p,N)^l_\alpha \in \mathfrak{S}$ there exists $i \in I_n$ such that $i_\alpha \leq i < i_{\alpha+1}$ such that $(p,N)^l_\alpha \in \mathfrak{S}t(M_i)$. So $p^l_\alpha \mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}^l_\alpha}_{N^l_\alpha} M_i$. Since by definition of strong type M_i is universal over N^l_α and $M_i \prec_{\mathcal{K}} M'_l$, by proposition 1.4.9 there exists $p^* \in \text{ga-S}(M'_l)$ which extends p^l_α and $p^* \mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}^l_\alpha}_{N^l_\alpha} M'_l$. Notice that $M'_{succ_{I_{n+1}}(l)}$ is universal over M'_l (by construction), then there exists $a_l \in M'_{succ_{I_{n+1}}(l)}$ such that $a_l \models p^*$. Consider $N_l := N^l_\alpha$. So, $a_l \mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}}_{N_l} M'_l$. The s-tower constructed in this way satisfies the requirements of the proposition. $\square_{\text{Lemma 2.2.21}}$

2.2.4 Uniqueness of Limit Models

The following fact is inspired by the related result given in [GVV08]. Although the sketch of the proof in the metric case is the same as the proof given in [GVV08], we have to point out that the details of the steps in the proof are quite different.

Proposition 2.2.22. *There is a* (μ,θ)-d-*limit model over* M *which is also a* (μ,ω)-d-*limit model over* M.

Proof. Consider a tower $(\mathfrak{M},\overline{a},\mathfrak{N},\mathcal{M},\mathcal{N})^0 \in \mathcal{K}_{\mu,I_0}$ such that $M^0_0 := M$. Suppose that we have constructed $(\mathfrak{M},\overline{a},\mathfrak{N},\mathcal{M},\mathcal{N})^n \in \mathcal{K}_{\mu,I_n}$. By lemma 2.2.21 and proposition 2.2.10, there exists an s-tower $(\mathfrak{M},\overline{a},\mathfrak{N},\mathcal{M},\mathcal{N})^n \leq (\mathfrak{M},\overline{a},\mathfrak{N},\mathcal{M},\mathcal{N})^{n+1}$ (witnessed by $f_{n,n+1} : \overline{\bigcup_{i\in I_n} M^n_i} \to \overline{\bigcup_{j\in I_{n+1}} M^{n+1}_j}$) which is reduced and also satisfies the properties given in lemma 2.2.21. At I_ω, consider $(\mathfrak{M},\overline{a},\mathfrak{N},\mathcal{M},\mathcal{N})^\omega$ as the directed limit of $\langle(\mathfrak{M},\overline{a},\mathfrak{N},\mathcal{M},\mathcal{N})^n, f_{n,m} : n \leq m < \omega\rangle$. By proposition 2.2.11, $(\mathfrak{M},\overline{a},\mathfrak{N},\mathcal{M},\mathcal{N})^\omega$ is a reduced tower (and so continuous, by proposition 2.2.13).

Claim 2.2.23. $M^\omega_{i_\theta}$ *is a* (μ,ω)-d-*limit model witnessed by* $\{f_{n,\omega}[M^n_{i_\theta}] : n < \omega\}$

Proof. By definition of $\leq$. $\square_{\text{Claim 2.2.23}}$

Claim 2.2.24. M^ω_θ *is a* (μ,θ)-d-*limit model*

Proof. $(\mathfrak{M},\overline{a},\mathfrak{N},\mathcal{M},\mathcal{N})^\omega$ is relatively full to $(f_{n,\omega}[M^n_i])_{n<\omega,i\in I_\omega}$ (by lemma 2.2.21). So, by proposition 2.2.19, $M^\omega_{i_\theta}$ is a (μ,θ)-d-limit witnessed by $\{M^\omega_i : i < \imath_\theta\}$ (notice that continuity of reduced towers guarantees that $M^\omega_{i_\theta} = \overline{\bigcup_{i<i_\theta} M^\omega_i}$). This finishes the proof of claim 2.2.24 $\square_{\text{Claim 2.2.24}}$

So, we have constructed a (μ,ω)-d-limit model over M which is also a (μ,θ)-d-limit model over M. $\square_{\text{Prop. 2.2.22}}$

Corollary 2.2.25. *If* M_i *is a* (μ,θ_i)-d-*limit over* M *(*$i \in \{1,2\}$*), then* $M_1 \approx_M M_2$.

Proof. By proposition 2.2.22, they are isomorphic to a (μ,ω)-d-limit model over M. $\square_{\text{Cor. 2.2.25}}$

CHAPTER 3

Domination, orthogonality and parallelism in superstable Metric Abstract Elementary Classes

The study of Zilber's trichotomy for strongly minimal sets in understanding the classification -up to bi-interpretability- of uncountably categorical strongly minimal theories is considered the beginning of Geometric Stability Theory, although restricted to ω_1-categoricity. However, some people consider that the actual birth of Geometric Stability Theory is the study of the non-finite axiomatizability of totally categorical theories -works of Cherlin, Harrington, Lachlan and Zilber-. Buechler used generalizations of this machinery outside of totally-categorical and ω_1-categorical settings and got a proof of his famous dichotomy theorem on the collection D of realizations of $\mathrm{stp}(a/A)$ for a any realization of a weakly minimal type, which says that either D is locally modular or p has Morley rank 1 [Bue85].

In Superstable First Order theories, there is a nice development of Geometric Stability Theory (see [Pil96, Bue96]), which corresponds to a generalization of the results studied in the categorical settings. In his doctoral thesis, E. Hrushovski extended this work to Stable First Order Theories [Hru86]. Also, this study has been extended to Rosy Theories by A. Onshuus and A. Usvyatsov (see [OU09a]). In abstract settings, S. Shelah provided some extensions of these results in *AEC Good Frames* (see [She09b]), which corresponds to a setting that J. Baldwin calls *intermediate stability theory* because it does not really consider more refined techniques of geometric stability theory, e.g. group configurations and Hrushovski's analysis.

This chapter is devoted to the study of some basic notions of classical Geometric Stability Theory -*domination, orthogonality and parallelism*, in fact notions well-behaved in stable theories- in *Superstable MAECs*, extending the results given by J. Baldwin ([Bal0x]) and Shelah [She09b], and studying some properties which were not proved in [She09b].

Assumption 3.0.26. *Throughout this chapter, we assume AP, JEP, CTP, existence of arbitrarily large enough models and 2.2.2 (the superstability-like assumptions which we used in chapter 2). For the* *sake of completeness, we write those assumptions once more: For every* a *and every increasing and continuous* $\prec_{\mathcal{K}}$*-chain of models* $\langle M_i : i < \sigma \rangle$ *and* M_j *a resolution of* M_j *(*$j < \sigma$*):*

1. *(Continuity) If* $p \upharpoonright M_i \downarrow_{M_0}^{\mathcal{M}_0} M_i$ *for all* $i < \sigma$*, then* $p \downarrow_{M_0}^{\mathcal{M}_0} \overline{\bigcup_{i<\sigma} M_i}$*.*

2. *(Locality) if* $cf(\sigma) > \omega$*, there exists* $j < \sigma$ *such that* $a \downarrow_{M_j}^{\mathcal{M}_j} \bigcup_{i<\sigma} M_i$*.*

3. *(*ε*-simplicity) if* $cf(\sigma) = \omega$*, there exists* $j < \sigma$ *such that* $a \downarrow_{M_j}^{\varepsilon} \overline{\bigcup_{i<\sigma} M_i}$*.*

Under these assumptions, we proved uniqueness of limit models (see corollary 2.2.25): If M_i is a (μ, θ_i)-d-limit over M ($i \in \{1, 2\}$) such that $dc(M_1) = dc(M_2)$, then $M_1 \approx_M M_2$. Remember that assumptions 2.2.2 2. and 3. imply $a \downarrow_M^{\mathcal{M}} M$ for every M and every resolution $\mathcal{M}$ of M (remark 2.2.4)

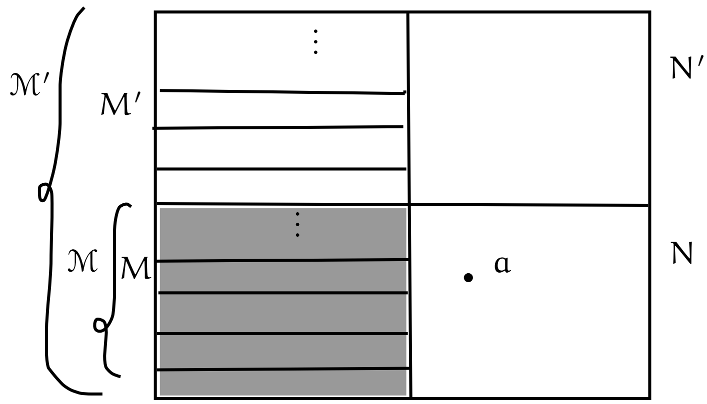
3.1 Domination in MAEC

In this section, we define a natural adaptation of the notion of domination in the setting of superstable MAECs that exhibit the superstability-like assumption 2.2.2. We base the development of this section on [Bal0x] but we use s-independence instead of intersections as Baldwin does.

According to S. Buechler ([Bue96]), the motivating question which takes us to the notion of *domination* is whether nonorthogonal (first order syntactical) types p and q have bases relative to a model M (i.e., maximal Morley sequences of p and q respectively over the domain of the respective types contained in M) with the same cardinality. In such context, domination is a kind of opposite notion to orthogonality. In first order, we say that a (possibly infinite) set B dominates another (possibly infinite) set A over C if and only if for any set D, if $B \downarrow_C D$ then $A \downarrow_C D$. But in our setting, we cannot define independence on sets because, in general, Galois types are defined on models. Because of that, we have to adapt this notion to our general context.

Notation 3.1.1. $(M, \mathcal{M}, N, a)$ means that $M \prec_{\mathcal{K}} N$, M is a limit model witnessed by $\mathcal{M}$ and $a \in N \setminus M$.

Definition 3.1.2. We say that $(M, \mathcal{M}, N, a) \prec_{nf} (M', \mathcal{M}', N', a)$ if and only if M' is a limit model over M, $\mathcal{M} \subset \mathcal{M}'$ and $\mathcal{M}$ corresponds to an initial segment of $\mathcal{M}'$, $N \prec_{\mathcal{K}} N'$ and $a \downarrow_M^{\mathcal{M}} M'$.



Definition 3.1.3. We say that a set A is smooth independent from M over N relative to a resolution $\mathcal{N}$ of N -denoted by $A\downarrow_N^{\mathcal{N}}M$ - if and only if $b\downarrow_N^{\mathcal{N}}M$ for every finite tuple $b \in A$.

Definition 3.1.4. Given $(M, \mathcal{M}, N, a)$, we say that a *dominates* N over M relative to $\mathcal{M}$ (denoted by $a \rhd_M^{\mathcal{M}} N$) iff for every $(M', \mathcal{M}', N', a) \succ_{nf} (M, \mathcal{M}, N, a)$ we have that $N\downarrow_M^{\mathcal{M}}M'$ (i.e., for every $b \in N$ $b\downarrow_M^{\mathcal{M}}M'$).

Remember that in first order, B dominates A over C if and only if for any set D, if $B\downarrow_C D$ then $A\downarrow_C D$. Because in our general context Galois types are defined on models instead of sets, we have to adapt this notion to our setting. Notice that $(M', \mathcal{M}', N', a) \succ_{nf} (M, \mathcal{M}, N, a)$ implies $a\downarrow_M^{\mathcal{M}}M'$, so $a \rhd_M^{\mathcal{M}} N$ means that $a\downarrow_M^{\mathcal{M}}M'$ implies $N\downarrow_M^{\mathcal{M}}M'$, agreeing with the first order notion of domination.
The following proposition says that domination over a model M_α implies domination over a $\mathcal{K}$-superstructure $M \succ_{\mathcal{K}} M_\alpha$ if there is some independence from M over M_α (i.e., the information given over M is the same over M_α).

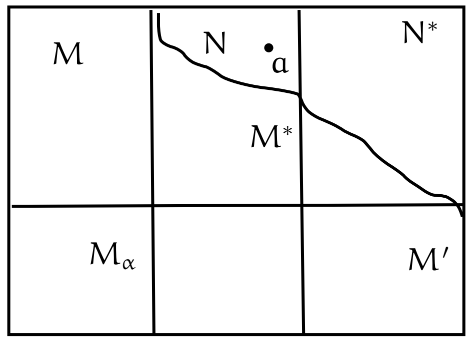
Proposition 3.1.5. *Let* $(M, \mathcal{M}, N, a)$ *(where* $\mathcal{M} := \{M_i : i < \theta\}$ *witnesses that* M *is a limit model) and* $\mathcal{M}_\alpha \subset \mathcal{M}$ *be a resolution of* M_α *(*$\alpha < \theta$*) such that* $a\downarrow_{M_\alpha}^{\mathcal{M}_\alpha}M$. *If* $a \rhd_{M_\alpha}^{\mathcal{M}_\alpha} N$ *then* $a \rhd_M^{\mathcal{M}} N$.

Proof. Let $(M', \mathcal{M}', N', a) \succ_{nf} (M, \mathcal{M}, N, a)$. Therefore, $a\downarrow_M^{\mathcal{M}}M'$. By hypothesis $a\downarrow_{M_\alpha}^{\mathcal{M}_\alpha}M$, hence $a\downarrow_{M_\alpha}^{\mathcal{M}_\alpha}M'$ (by transitivity, proposition 1.4.19). So, $(M', \mathcal{M}', N', a) \succ_{nf} (M_\alpha, \mathcal{M}_\alpha, N, a)$. Since $a \rhd_{M_\alpha}^{\mathcal{M}_\alpha} N$, then $N\downarrow_{M_\alpha}^{\mathcal{M}_\alpha}M'$. By monotonicity (proposition 1.4.6), $N\downarrow_M^{\mathcal{M}}M'$, therefore $a \rhd_M^{\mathcal{M}} N$. $\square_{\text{Prop. 3.1.5}}$

The following proposition is a kind of reciprocal of proposition 3.1.5. This says that under some independence from M over M_α, domination over M implies domination over M_α.

Proposition 3.1.6. *Let* $(M, \mathcal{M}, N, a)$ *(where* $\mathcal{M} := \{M_i : i < \theta\}$ *witnesses that* M *is a* (μ, σ)*-limit model) and* $\mathcal{M}_\alpha \subset \mathcal{M}$ *be a resolution of* M_α *(*$\alpha < \theta$*) such that* $N\downarrow_{M_\alpha}^{\mathcal{M}_\alpha}M$. *If* $a \rhd_M^{\mathcal{M}} N$ *then* $a \rhd_{M_\alpha}^{\mathcal{M}_\alpha} N$.

Proof. Let $(M', \mathcal{M}', N', a) \succ_{nf} (M_\alpha, \mathcal{M}_\alpha, N, a)$. Let $M \cup M' \subset \hat{M} \prec_{\mathcal{K}} \mathbb{M}$ (by downward Löwenheim-Skolem axiom) and $M^* \succ_{\mathcal{K}} \hat{M}$ be a limit over $\hat{M}$ -and so M^* is a limit model over M, where $\mathcal{M}^{**}$ is a witness of that-. Let $N^* \succ_{\mathcal{K}} N$ be such that $N^* \succ_{\mathcal{K}} M^*$. and $\mathcal{M}^* := \mathcal{M}^\frown \mathcal{M}^{**}$



Since $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M'$ (by definition of $\prec_{nf}$) and M' is universal over M_α, by the extension property of smooth independence (proposition 1.4.9), there exists $a' \models \text{ga-tp}(a/M')$ such that $a' \perp^{\mathcal{M}_\alpha}_{M_\alpha} M^*$. Without less of generality, suppose $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M^*$. Notice that $(M^*, \mathcal{M}^*, N^*, a) \succ_{nf} (M, \mathcal{M}, N, a)$. Since $a \rhd^{\mathcal{M}}_{M} N$, then $N \perp_{M} M^*$. By hypothesis, $N \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$, so by transitivity (proposition 1.4.19, since M and M_α are limit models over M_0) $N \perp_{M_\alpha} M^*$, and by monotonicity (proposition 1.4.6) $N \perp^{\mathcal{M}_\alpha}_{M_\alpha} M'$ (since $M_\alpha \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} M^*$). So, we have that $a \rhd^{\mathcal{M}_\alpha}_{M_\alpha} N$. $\square_{\text{Prop. 3.1.6}}$

The following proposition says that given any tuple $(M, \mathcal{M}, N, a)$, we can find some extensions $N' \succ_{\mathcal{K}} N$ and $M' \succ_{\mathcal{K}} M$ such that $N' \succ_{\mathcal{K}} M'$, and a dominates N' over M'.

Proposition 3.1.7. *Given* $(M, \mathcal{M}, N, a)$ *there exists* $(M', \mathcal{M}', N', a) \succ_{nf} (M, \mathcal{M}, N, a)$ *such that* $a \rhd^{\mathcal{M}'}_{M'} N'$.

Proof. Suppose not. This allows us to construct an $\prec_{nf}$-increasing and continuous sequence of tuples $\langle (M^\alpha, \mathcal{M}^\alpha, N^\alpha, a) : \alpha < \mu^+ \rangle$ such that $(M^0, \mathcal{M}^0, N^0, a) := (M, \mathcal{M}, N, a)$ and $(M^{\alpha+1}, \mathcal{M}^{\alpha+1}, N^{\alpha+1}, a)$ witnesses that $(M^\alpha, \mathcal{M}^\alpha, N^\alpha, a)$ does not satisfy that $a \rhd^{\mathcal{M}^\alpha}_{M^\alpha} N^\alpha$. Therefore, there exists $b \in N^\alpha$ such that $a \perp^{\mathcal{M}^\alpha}_{M^\alpha} M^{\alpha+1}$ but $b \not\perp^{\mathcal{M}^\alpha}_{M^\alpha} M^{\alpha+1}$. By assumption 2.2.2, given any c there exists $\alpha_c < \mu^+$ such that $c \perp^{\mathcal{M}^{\alpha_c}}_{M^{\alpha_c}} \bigcup_{\alpha<\mu^+} M^\alpha$.

Consider $\gamma_0 < \mu^+$. Since N^{γ_0} has density character μ, there exists B_{γ_0} a dense subset of N^{γ_0} of cardinality μ. Defining $f_0 : B_{\gamma_0} \to \mu^+$ as $f(c) := \alpha_c$, we have that there exists $\gamma_0' < \mu^+$ such that $f(c) := \alpha_c < \gamma_0'$ for every $c \in B_{\gamma_0}$. Define $\gamma_1 := \max\{\gamma_0, \gamma_0'\} + 1$.

In the same way we define B_{γ_n} and γ_n for every $n < \omega$. Notice that $(\gamma_n : n < \omega)$ is an increasing sequence of ordinals $< \mu^+$.

Define $\gamma := \sup\{\gamma_n : n < \omega\}$. Notice that $\gamma < \mu^+$.

Let $b \in N^\gamma$ be such that $b \not\perp^{\mathcal{M}^\gamma}_{M^\gamma} M^{\gamma+1}$. Since $N^\gamma := \overline{\bigcup_{\alpha<\gamma} N^\alpha}$, there exists a sequence $(b_n) \in \bigcup_{\alpha<\gamma} N^\alpha$ such that $(b_n) \to b$. By proposition 1.4.16 (continuity of $\perp$), there exists $k < \omega$ such that $b_k \not\perp^{\mathcal{M}^\gamma}_{M^\gamma} M^{\gamma+1}$. Since $b_k \in \bigcup_{\alpha<\gamma} N^\alpha$, there exists $\beta < \gamma$ such that $b_k \in N^\beta$. Since $\beta < \gamma := \sup\{\gamma_n : n < \omega\}$, there exists $m < \omega$ such that $\beta < \gamma_m$, so $b_k \in N^{\gamma_m}$. Since by construction we have that $\overline{B_{\gamma_m}} = N^{\gamma_m}$, there exists a sequence $(c_n) \in B_{\gamma_m}$ such that $(c_n) \to b_k$. By proposition 1.4.16 again, there exists $l < \omega$ such that $c := c_l \not\perp^{\mathcal{M}^\gamma}_{M^\gamma} M^{\gamma+1}$. By construction, $\alpha_c < \gamma_{m+1} < \gamma < \gamma + 1 < \mu^+$, then by proposition 1.4.6 (monotonicity of $\perp$) we have that $c \not\perp^{\mathcal{M}^{\alpha_c}}_{M^{\alpha_c}} \bigcup_{\alpha<\mu^+} M^\alpha$ (contradiction). Therefore, the proposition is true. $\square_{\text{Prop. 3.1.7}}$

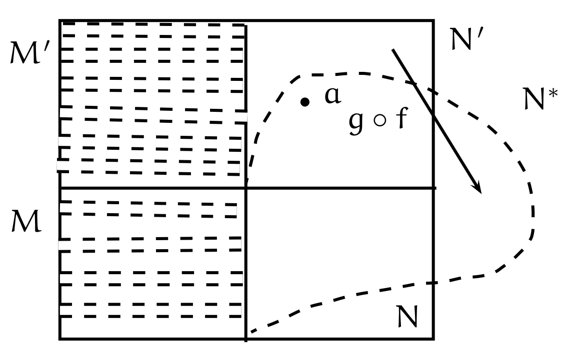
The following proposition says that under the conclusions of the previous proposition, we can find an extension N^* of N' such that a dominates N^* over M.

Proposition 3.1.8. *Suppose* $(M, \mathcal{M}, N, a) \prec_{nf} (M', \mathcal{M}', N', a)$, *where* M *is a* (μ, σ_1)*-limit model witnessed by* $\mathcal{M} := \langle M_i : i < \sigma_1 \rangle$, M' *is a* (μ, σ_2)*-limit model over* M *witnessed by* $\mathcal{M}''$ *and* $\mathcal{M}' := \mathcal{M}^\frown\mathcal{M}'$, $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$ *for some limit* $\alpha < \sigma$ *and* $a \rhd^{\mathcal{M}'}_{M'} N'$. *Then, there exist* N^* *and a resolution* $\mathcal{M}^*$ *which witnesses that* M *is a limit model over* M_0 *such that* $a \rhd^{\mathcal{M}^*}_{M} N^*$.

Proof. Let $p := \text{ga-tp}(a/M)$ and $p' := \text{ga-tp}(a/M')$. Since $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$ (by hypothesis), $a \perp^{\mathcal{M}}_{M} M'$ (by definition of $\prec_{nf}$) and M, M_α are limit models over M_0, by transitivity (proposition 1.4.19) we have $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M'$.
Notice that since M and M' are limit over M_α witnessed by $\mathcal{M}$ and $\mathcal{M}'$ respectively such and $\mathcal{M} \subset \mathcal{M}'$, then M and M' are limit over $M_1 \in \mathcal{M}$. By corollary 2.2.25 (uniqueness of limit models), there exists $f : M' \xrightarrow{\approx}_{M_{\alpha+1}} M$. Since $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M'$, we have that $f(a) \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$ (by invariance, proposition 1.4.5). Notice that $M_{\alpha+1}$ is universal over M_α. Then, as $\text{ga-tp}(a/M_{\alpha+1}) = \text{ga-tp}(f(a)/M_{\alpha+1})$ and $a, f(a) \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$, by stationarity (proposition 1.4.11) we may say $\text{ga-tp}(a/M) = \text{ga-tp}(f(a)/M)$.
Consider $g \in Aut(\mathbb{M}/M)$ such that $(g \circ f)(a) = a$. Notice that

$$(g \circ f)(M', \mathcal{M}', N', a) = (M, (g \circ f)[\mathcal{M}'], (g \circ f)[N'], a)$$

witnesses that $a \rhd^{\mathcal{M}^*}_{M} N^*$, where $N^* := (g \circ f)[N']$ and $\mathcal{M}^* := (g \circ f)[\mathcal{M}'] = f[\mathcal{M}']$. Notice that $\mathcal{M}^*$ is also a resolution which witnesses that M is a limit model over M_0 (remember that in particular f fixes M_0 pointwise).



$\square_{\text{Prop. 3.1.8}}$

Remark 3.1.9. Notice that given $(M, \mathcal{M}, \overline{a}, N)$, if M' is limit model over M such that $N \downarrow^{\mathcal{M}}_{M} M'$, in particular we have that $a \downarrow^{\mathcal{M}}_{M} M'$ because $a \in N$. Therefore, if $\overline{a} \triangleright^{\mathcal{M}}_{M} N$ we may say that $\overline{a}$ and N are equidominant over M relative to $\mathcal{M}$, which we denote by $\overline{a} \bowtie^{\mathcal{M}}_{M} N$.

Corollary 3.1.10. *Given $(M, \mathcal{M}, \overline{a}, N)$ such that $\overline{a} \downarrow^{\mathcal{M}_\alpha}_{M_\alpha} M$ for some limit ordinal α such that $M_\alpha \in \mathcal{M}$ (and therefore* ga-tp$(\overline{a}/M)$ *is a stationary type because M is an universal model over M_α), there exist N^* and a resolution $\mathcal{M}^*$ which witnesses that M is a limit model over M_0 such that $\overline{a} \bowtie^{\mathcal{M}^*}_{M} N^*$.*

Question 3.1.11. In general, we cannot assure the existence of prime models in metric and discrete AECs. In superstable first order theories, we can prove that if p is a stationary syntactic type, there exist regular types $p_1, \cdots, p_n$ such that $p \bowtie p_1 \otimes \cdots \otimes p_n$. Setting $(\overline{a}_1, \cdots, \overline{a}_n) \models p_1 \otimes \cdots \otimes p_n$ and $\overline{a} \models p$, it is known that $M[\overline{a}_1, \cdots \overline{a}_n] = M[\overline{a}]$ (i.e., the a-prime model over $M \cup \{\overline{a}\}$ and the a-prime model over $M \cup \{\overline{a}_1, \cdots, \overline{a}_n\}$ agree). In Hilbert spaces with a unitary operator (section 5.1), notice that corollary 3.1.10 says that that given $\overline{a} \in \mathcal{H}$ (where $\mathcal{H}$ is a monster Hilbert space with a unitary operator) and M a Hilbert space with a unitary operator such that M is saturated enough and $\overline{a} \notin M$, there exists a Hilbert space with a unitary operator $N^* \supset acl(M\overline{a})$ extending M such that for every Hilbert space with a unitary operator $M' \geq M$, $P_M(\overline{a}) = P_{M'}(\overline{a})$ (the orthogonal projections over M and M' respectively) implies that $P_M(b) = P_{M'}(b)$ for every $b \in N^*$; i.e.: $\overline{a}$ determines the projections on M of all elements in N^*. A natural question that arises at this point is under which assumptions (e.g., ω-d-stability or related properties) an MAEC admits prime models over sets. This question takes us to wonder what we can conclude on domination, orthogonality and parallelism under existence of primer models over sets.

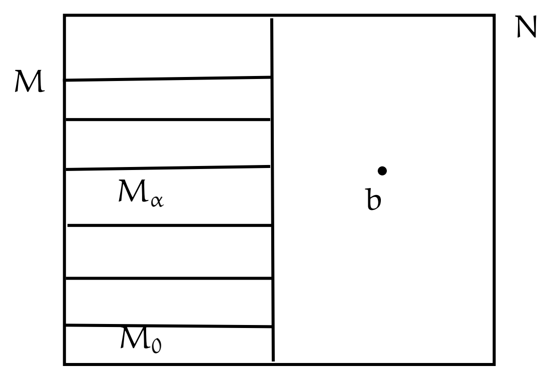
3.2 Orthogonality

Orthogonality arose from the question on the existence of bases -maximal Morley sequences- of arbitrary size in a model, for (first order syntactical types) p and q (see [Bue96]).

In this section, we adapt the study of orthogonality which S. Shelah did in the setting of good frames in (discrete) Abstract Elementary Classes (see [She09b, She09a]). Shelah provided a suitable study of superstability in (discrete) AECs via good frames. But he did not assume the existence of a monster model as in 1.3.5 and he worked on an abstract notion of independence. Most of the definitions in this section are inspired on Shelah's work ([She09b]), with some exceptions (e.g., the definition of domination of types, which we define in this thesis in order to prove that domination corresponds to a kind of nonorthogonality), but we have to point out that our analysis is a bit different than the work done in [She09b] because we are assuming the existence of a homogeneous monster model and we are working on a fixed notion of independence (smooth independence). In this section, we obtain a "simplification" of the notions given by Shelah to our setting and prove some basic facts which were not proved in [She09b].

Although, we have to point out that there would be some problems proving the existence of weakly orthogonal types in the way as our notion of weak orthogonality is being defined. In spite of that, we develop this section for showing some nice properties as a consequence of uniqueness of limit models.

Notation 3.2.1. $(M, \mathcal{M}, N, b, \alpha)$ means $\mathcal{M} := \{M_i : i < \delta\}$ is a resolution of M which witnesses that M is a limit model, $\alpha < \delta$ is a limit ordinal, $M \prec_{\mathcal{K}} N$, $b \in N \setminus M$ and $b \mathop{\smile\!\!\!\!\!\!\;|}_{M_\alpha}^{\mathcal{M}_\alpha} M$, where $\mathcal{M}_\alpha$ is a resolution of M_α such that $\mathcal{M}_\alpha \subset \mathcal{M}$.

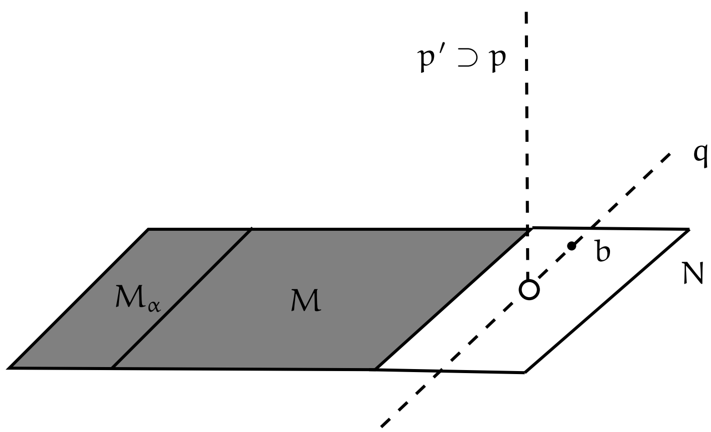


3.2.1 Orthogonality and Independence of sequences

Definition 3.2.2. Let $\mathbb{J}$ be a sequence of elements in $\mathbb{M}$, $M \subset N$ where $M \prec_{\mathcal{K}} N$ and $\mathcal{M}$ be a resolution of M. We say that $\mathbb{J}$ *is independent* in (M, N) iff there exist $\langle M_j, a_i : j \leq \alpha, i < \alpha \rangle$ and $N^+ \in \mathcal{K}$ such that

1. $\langle N_i : i \leq \alpha \rangle$ is a $\prec_{\mathcal{K}}$-increasing and continuous chain.
2. $\mathbb{J} = \{a_i : i < \alpha\}$.
3. $M \prec_{\mathcal{K}} N_i \prec_{\mathcal{K}} N^+$ and $N \prec_{\mathcal{K}} N^+$.
4. $a_i \in N_{i+1} \setminus N_i$.
5. $a_i \mathop{\smile\!\!\!\!\!\!\;|}_{M}^{\mathcal{M}} N_i$.

Definition 3.2.3. Let $M \in \mathcal{K}$ be a limit model witnessed by $\mathcal{M}$ and $\mathcal{M}_\alpha$ be a resolution of a model $M_\alpha \in \mathcal{M}$ such that $\mathcal{M}_\alpha \subset \mathcal{M}$. Let $p, q \in$ ga-S(M) be non-algebraic types such that $p, q \mathop{\smile\!\!\!\!\!\!\;|}_{M_\alpha}^{\mathcal{M}_\alpha} M$. We say that p *is weakly orthogonal to* q *relative to* α (denoted by $p \perp_\alpha^{wk} q$) iff given $(M, \mathcal{M}, N, b, \alpha)$ where $b \models q$ and $p' \in$ ga-S(N) any extension of p, then $p' \mathop{\smile\!\!\!\!\!\!\;|}_{M}^{\mathcal{M}} N$ -notice that by definition of $(M, \mathcal{M}, N, b, \alpha)$, $b \in N$-. We drop the subindex α if it is clear.

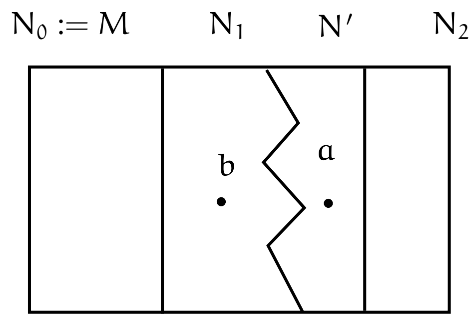


In first order, for stationary types $p, q \in S(A)$, we say that p is (almost) orthogonal to q (over A) if and only if there exist realizations $a \models p$ and $b \models q$ such that $a \mathop{\smile\!\!\!\!\!\!|}_{A} b$. Since in our setting we cannot consider independence either from or over sets which are not models, we have to adapt this notion to our context, as in [She09b]. Notice that if we could define smooth independence on sets, $p' \mathop{\smile\!\!\!\!\!\!|}_{M}^{\mathcal{M}} N$ would imply $p \mathop{\smile\!\!\!\!\!\!|}_{M}^{\mathcal{M}} b$ because $b \in N$. Hence, weak orthogonality corresponds to a stronger notion of orthogonality. In spite of that, the notion of independence defined at the beginning of this subsection (independence of sequences, definition 3.2.2) allows us to catch such independence between a realization a of p and a realization b of q.

Example 3.2.4. Consider the class of Hilbert spaces. As in Hilbert spaces together with a unitary operator (see section 5.1), independence is characterized by agreeing with the respective projections (i.e., $a \mathop{\smile\!\!\!\!\!\!|}_{N} M$ if and only if $P_N(a) = P_M(a)$). In this case, $p \perp^{wk} q$ (both of them in ga-$S(M)$) means that for every Hilbert space $N \geq M$ which contains a realization of q and given any realization a of p, $P_M(a) = P_N(a)$. If $M = \langle 0 \rangle$ and $N = \langle b \rangle$, notice that weak orthogonality implies that $0 = P_M(a) = P_N(a)$, therefore a and b are orthogonal in the sense of the inner product in Hilbert spaces.

Proposition 3.2.5. *Let* $p, q \in$ ga-$S(M)$ *be such that* $p \perp_{\alpha}^{wk} q$ *and* $\mathcal{M}_\alpha$ *and* $\mathcal{M}$ *be resolutions of* M_α *and* M *respectively such that* $\mathcal{M}_\alpha \subset \mathcal{M}$*, where* $\mathcal{M}$ *witnesses that* M *is a limit model. If* $N' \succ_{\mathcal{K}} M$ *contains a realization* a *of* p *and a realization* b *of* q*, then* $\{a, b\}$ *is independent in* (M, N')*.*

Proof. Define $N_0 := M$. Given M, N', and $a, b \in N'$ as above, by Downward Löwenheim-Skolem axiom of MAEC (definition 1.1.4 (6)) there exists $N_1 \in \mathcal{K}$ of size μ such that $N_0 \cup \{b\} \subset N_1 \prec_{\mathcal{K}} N'$. Notice that $b \notin N_0$ (since q is non-algebraic). Trivially we have that $b \mathop{\smile\!\!\!\!\!\!|}_{N_0}^{\mathcal{N}_0} N_0$. Since ga-tp$(a/N_1) \supset p$, applying $p \perp_{\alpha}^{wk} q$ to $(M, \mathcal{M}, N_1, b, \alpha)$ we may say $a \mathop{\smile\!\!\!\!\!\!|}_{N_0}^{\mathcal{N}_0} N_1$. Notice that $a \notin N_1$ since $a \notin N_0$ (since p is non-algebraic and by antirreflexivity, proposition 1.4.12). Let $N_2 \succ_{\mathcal{K}} N'$.



Defining $N^+ := N_2$, $a_0 := b$ and $a_1 := a$, notice that $\{N_0, N_1, N_2; a_0, a_1\}$ and N^+ witness that $\{a, b\}$ is independent in (M, N'). $\square_{\text{Prop. 3.2.5}}$

The following proposition says that given $p, q \in$ ga-S$(M))$ and $N \succ_{\mathcal{K}} M$ has a realization of q, then p is weakly orthogonal to q if and only if p has just one extension in ga-S(N).

Proposition 3.2.6. *Let* $p, q \in$ ga-S(M) *be non-algebraic types,* $\mathcal{M}$ *be a resolution of* M *which witnesses that* M *is a limit model such that* $p, q \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}_\alpha}_{M_\alpha} M$ *where* $M_\alpha \in \mathcal{M}$ *and* $\mathcal{M}_\alpha \subset \mathcal{M}$ *is a resolution of* M_α*. Then* $p \perp^{wk}_{\alpha} q \Leftrightarrow$ *for any* $(M, \mathcal{M}, N, b, \alpha)$ *such that* $q =$ ga-tp(b/M)*,* p *has just one extension in* ga-S(N).

Proof.

- $\Rightarrow$ Suppose $p \perp^{wk}_{\alpha} q$, then every extension $p' \in$ ga-S(N) of p satisfies $p' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}}_{M} N$; since M_α and M are limit model over M_0, by transitivity (proposition 1.4.19) we have $p' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}_\alpha}_{M_\alpha} N$. By stationarity over limit models of s-independence (proposition 1.4.11), we have that there is just one extension of p in ga-S(N).
- $\Leftarrow$ Let $(M, \mathcal{M}, b, N, \alpha)$ be such that $b \models q$ and suppose p has just one extension in ga-S(N). By extension property of s-independence (proposition 1.4.9, since M is a limit model over M_α), there exists an extension $p' \supset p$ in ga-S(N) such that $p' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}_\alpha}_{M_\alpha} N$. By monotonicity (proposition 1.4.6 , since $\mathcal{M}_\alpha \subset \mathcal{M}$), $p' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}}_{M} N$. Since p has just one extension in ga-S(N), then the unique extension of p in ga-S(N) is p'. Therefore, $p \perp^{wk}_{\alpha} q$.

$\square_{\text{Prop. 3.2.6}}$

Proposition 3.2.7. *Let* $(M, \mathcal{M}, N, b, \alpha)$ *be such that* $b \models q$ *and* $p \in$ ga-S(M) *be a non-algebraic Galois type such that* $p \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}_\alpha}_{M_\alpha} M$*. If* p *is realized in* N*, then* $p \not\perp^{wk}_{\alpha} q$.

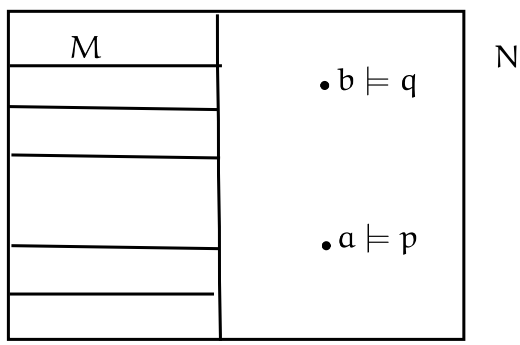
Proof. Since M is universal over M_α and $p \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}_\alpha}_{M_\alpha} M$, by extension property (proposition 1.4.9) there exists $p' \supset p$ in ga-S(N) such that $p' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}_\alpha}_{M_\alpha} N$. Notice that p' is non-algebraic: otherwise, by antirreflexivity (proposition 1.4.12 (6), since M_α is a limit model witnessed by $\mathcal{M}_\alpha$) p' would be realized in M_α and so realized in M (contradiction). But by hypothesis, there exists $c \in N$ such that $c \models p$. Notice that $p'' :=$ ga-tp$(c/N) \supset p$ and $p' \neq p''$ (contradicts fact 3.2.6).$\square_{\text{Prop. 3.2.7}}$

Another way to understand the previous proposition is the following corollary:

Corollary 3.2.8. *Let* $(M, \mathcal{M}, N, b, \alpha)$ *be such that* $b \models q$ *and* $p \in \text{ga-S}(M)$ *be a non-algebraic Galois type such that* $p \mathop{\smile\!\!\!\!\!\!\;|}_{M_\alpha}^{\mathcal{M}_\alpha} M$*. If* $p \perp_\alpha^{wk} q$*, then* p *is not realized in* N*.*

As we stated in section 3.1, orthogonality corresponds (in first order) in some way to nonorthogonality. In order to prove a similar result in our context, we adapt the notion of domination of types.

Definition 3.2.9. Let $p, q \in \text{ga-S}(M)$ be non-algebraic Galois types such that $p, q \mathop{\smile\!\!\!\!\!\!\;|}_{M_\alpha}^{\mathcal{M}_\alpha} M$. We say that q *is dominated by* p (denoted by $q \lhd p$) if there exist $a \models p$, $b \models q$ and $N \succ_{\mathcal{K}} M$ such that $a, b \in N$ and $N \lhd_M a$.



The following propositions says that domination of types implies non-weak orthogonality.

Proposition 3.2.10. *Let* $\mathcal{M} := \{M_i : i < \theta\}$ *be a resolution of a model* M *and* $\mathcal{M}_\alpha \subset \mathcal{M}$ *be a resolution of* M_α*. If* $q \lhd p$ *then* $p \not\perp_\alpha^{wk} q$ *and* $q \not\perp_\alpha^{wk} p$*.*

Proof. Let $a \models p$, $b \models q$ and $N \prec_{\mathcal{K}} M$ be witnesses of $q \lhd p$. Notice that $p' := \text{ga-tp}(a/N)$ is an extension of p such that $a \not\mathop{\smile\!\!\!\!\!\!\;|}_M^{\mathcal{M}} N$ (by antirreflexivity, since $a \in N \setminus M$). Notice that $(M, \mathcal{M}, N, b, \alpha)$ witnesses that $p \not\perp_\alpha^{wk} q$. By an analogous argument, we can prove $q \not\perp_\alpha^{wk} p$, using $(M, \mathcal{M}, N, a, \alpha)$. $\square_{\text{Prop. 3.2.10}}$

3.3 Parallelism

Roughly speaking, two (first order syntactical) stationary types p and q are parallel if and only if they have a common independent extension. In this section, we study parallelism of strong limit Galois types in the setting of superstable MAECs.

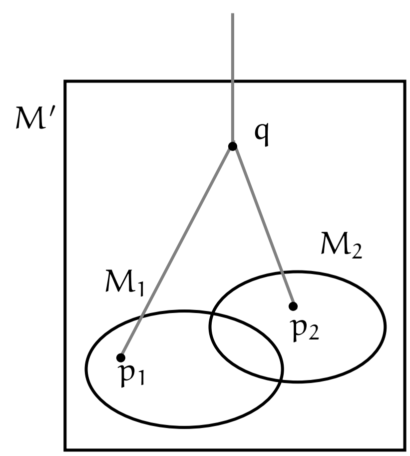
Remember that we defined parallelism of strong Galois types in chapter 2 (definition 2.2.15 (2)), which we used it as an auxiliary tool for studying full-relative s-towers. Full-relative s-towers were very important to get a proof of uniqueness of limit models because they codified a kind of saturation. In this subsection, we study some properties of a stronger version of parallelism, but in the setting of superstable MAECs.

However, for the sake of completeness, we provide the definition of parallelism once more. But we have to point out that in this subsection, we require that if $(p, N) \in \mathfrak{St}(M)$, then M is a limit model over N, instead of just being a universal model over N. Because of that, we define a stronger notion of strong type, which we call *strong limit type*. In this thesis, we use the notion of *strong limit type*instead of *strong types* because we want to use *uniqueness of limit models* to prove some properties of parallelism, e.g. proposition 3.3.6 (2) and (3).

Definition 3.3.1 (strong limit type). Let M be a (μ, σ)-limit model

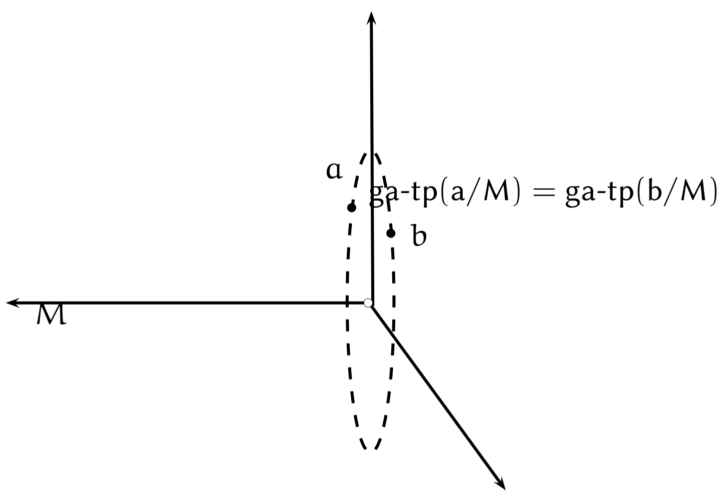
$$\mathfrak{SL}(M) := \left\{ (p, N) : \begin{array}{l} N \prec_{\mathcal{K}} M \\ N \text{ is a } \theta\text{-Limit Model} \\ M \text{ is a Limit Model over } N \\ p \in \text{ga-S}(M) \text{ is non-algebraic} \\ \text{and } p \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathcal{N}}_{N} M \\ \text{for some resolution } \mathcal{N} \text{ of } N. \end{array} \right\}$$

Definition 3.3.2 (Parallelism). Two strong limit types $(p_l, N_l) \in \mathfrak{SL}(M_l)$ ($l \in \{1, 2\}$) are said to be *parallel* (which we denote by $(p_1, N_1) \parallel (p_2, N_2)$) iff for every $M' \succ_{\mathcal{K}} M_1, M_2$ with density character μ, there exists $q \in \text{ga-S}(M')$ which extends both p_1 and p_2 and $q \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathcal{N}_l}_{N_l} M'$ ($l \in \{1, 2\}$) (where $\mathcal{N}_l$ is the resolution of N_l which satisfies $p_l \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathcal{N}_l}_{N_l} M_l$). If there is no any confusion, we denote it by $p_1 \parallel p_2$.

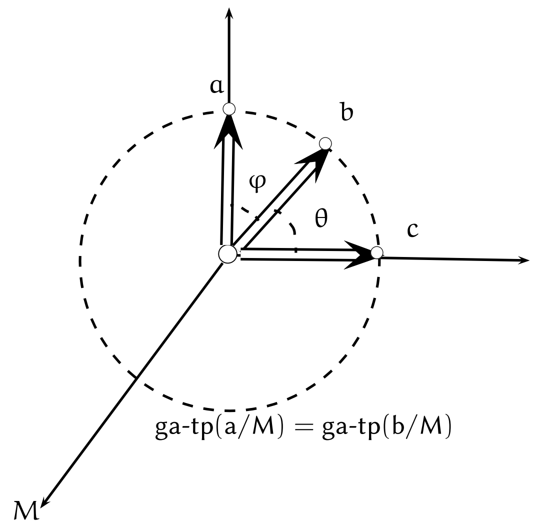


Remark 3.3.3. Consider the class of Hilbert spaces. Let us suppose that we could set $N_1 = N_2 = \langle 0 \rangle \subset \mathbb{R}^3$ -the space generated by the origin- (despite this is not a universal model) and let $M := M_1 = M_2 = \{(x, 0, 0) : x \in \mathbb{R}\}$. Remember that we stated in 3.2.4 that, as in a Hilbert space with a unitary operator (see section 5.1), independence in Hilbert spaces means the

respective projections agree. Let $p_i \in \text{ga-S}(M)$, $a \models p_1$ and $b \models p_2$ be such that a and b are independent from M over $\langle 0 \rangle$; i.e.: $0 = P_M(a) = P_M(b)$, therefore a and b are orthogonal to M.



If p_1 and p_2 are parallel in the sense defined above, consider $M' := \langle M \cup \{a, b\} \rangle \geq M$, so there exists a type over M' $q \supset p_1, p_2$ (so $\text{ga-tp}(a/M) = \text{ga-tp}(b)/M) = q \upharpoonright M$) such that q is independent from M' over $\langle 0 \rangle$ (i.e., any realization $c \models q$ satisfies $0 = P_{M'}(c)$), therefore c is orthogonal to a and b. If $c \in \mathbb{R}^3$, notice that it means that if φ is the angle between a and b and θ is the angle between b and c (and so $\theta + \varphi$ is the angle between a and c), since a and c are orthogonal then $|\cos(\varphi + \theta)| = 0$, and since b and c are orthogonal then $|\cos(\theta)| = \cos(\theta) = 0$, then $\theta = \frac{(2k+1)\pi}{2}$ for some $k \in \mathbb{Z}$. Since $0 = |\cos(\varphi + \theta)| = |\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)| = |\sin\left(\frac{(2k+1)\pi}{2}\right)\sin(\varphi)| = |\sin(\varphi)|$, therefore $\varphi = m\pi$ for some $m \in \mathbb{Z}$; i.e.: a and b would be parallel as vectors in $\mathbb{R}^3$.



Claim 3.3.4. $\|$ *is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial. We focus on transitivity. Let $(p_l, N_l) \in \mathfrak{S}\mathfrak{L}(M_l)$ ($l \in \{1,2,3\}$) be such that $p_1 \parallel p_2$ and $p_2 \parallel p_3$. Let M be a $\mathcal{K}$-extension of both M_1 and M_3. By Downward Löwenheim-Skolem axiom (Definition 1.1.4 (6)) and Coherence axiom of MAEC (Definition 1.1.4 (5)), there exists a $\mathcal{K}$-extension M' of both M and M_2. Denote by $p_l^{M'}$ ($l \in \{1,2,3\}$) the unique s-independent extension of p_l in ga-S(M') (such that extension exists by propositions 1.4.9 and 1.4.11, since M is universal over N_l) and by p_k^M ($k \in \{1,3\}$) the unique s-independent extension of p_k in ga-S(M). Notice that for $k \in \{1,3\}$ we have $p_k^M = p_k^{M'} \upharpoonright M$. Since $M_1, M_2 \prec_{\mathcal{K}} M'$ and $p_1 \parallel p_2$, then $p_1^{M'} = p_2^{M'}$. In a similar way we have $p_2^{M'} = p_3^{M'}$, then $p_1^{M'} = p_3^{M'}$. Since $M \prec_{\mathcal{K}} M'$, then $p_1^M = p_1^{M'} \upharpoonright M = p_3^{M'} \upharpoonright M = p_3^M$. Therefore, $p_1 \parallel p_3$. $\square_{\text{Claim 3.3.4}}$

The following proposition says that strong limit types are stationary (up to parallelism).

Proposition 3.3.5 ("Stationarity" of parallelism). *Let* $(p, N) \in \mathfrak{S}\mathfrak{L}(M)$ *and* $M' \succ_{\mathcal{K}} M$ *be a limit model over* M. *There exists a unique* $(q, N) \in \mathfrak{S}\mathfrak{L}(M')$ *such that* $p \parallel q$.

Proof. Since M is universal over N, by stationarity (proposition 1.4.11) there exists a unique $q \in$ ga-S(M') such that $q \mathop{\smile\!\!\!\!\!|}\,^{\mathcal{N}}_{N} M'$. Notice that $(q, N) \in \mathfrak{S}\mathfrak{L}(M')$.
We have that $p \parallel q$: Let $M'' \succ_{\mathcal{K}} M'$ (and so $M'' \succ_{\mathcal{K}} M$). If $p', q' \in$ ga-S(M') are the s-independent extensions of p and q respectively, we have $p' = q'$ (if not, $p' \neq q'$ are s-independent extensions of p, contradicts stationarity). Therefore $p \parallel q$.
If $(q^*, N) \in \mathfrak{S}\mathfrak{L}(M')$ satisfies $p \parallel q^*$ and q' $\in$ ga-S(M') is the unique extension of p and q^* (so $q^* = q'$) such that $q' \mathop{\smile\!\!\!\!\!|}\,^{\mathcal{N}}_{N} M'$, then by stationarity (proposition 1.4.11) $q^* = q' = q$. So, uniqueness is proved. $\square_{\text{Prop. 3.3.5}}$

Next, we prove that weak orthogonality is preserved under parallelism. Before giving its proof, we prove that weak orthogonality is invariant under isomorphisms and that weak orthogonality is preserved under $\mathcal{K}$-submodels and $\mathcal{K}$-superstructures if we have suitable independence conditions.

Proposition 3.3.6. *1. Given* $p, q \in$ ga-S(M), $\mathcal{M} := \{M_i : i < \delta\}$ *a resolution of* M *which witnesses that* M *is a limit model such that* $p, q \downarrow_{M_\alpha}^{\mathcal{M}_\alpha} M$ *for some* $\alpha < \delta$ *(where* $\mathcal{M}_\alpha \subset \mathcal{M}$ *is a resolution of* M_α*) and* $f : M \approx N$ *is an isomorphism, then* $p \perp_\alpha^{wk} q \Leftrightarrow f(p) \perp_\alpha^{wk} f(q)$.

2. *Given* $\mathcal{M} := \{M_i : i < \delta\}$ *a resolution which witnesses that* M *is a limit model, if* $N \succ_{\mathcal{K}} M$ *is limit over* M*, given* $p, q \in$ ga-S(N) *such that* $p, q \downarrow_{M_\alpha}^{\mathcal{M}_\alpha} N$, $p \perp_\alpha^{wk} q \Leftrightarrow p \upharpoonright M \perp_\alpha^{wk} q \upharpoonright M$.

3. *Given* $\mathcal{M} := \{M_i : i < \delta\}$ *a resolution which witnesses that* M *is a limit model, if* $N \succ_{\mathcal{K}} M$ *is a limit model over* M *(and in particular over* $M_{\alpha+1}$*) and* $(p_l, M_\alpha) \in \mathfrak{SL}(M)$ *and* $(q_l, M_\alpha) \in \mathfrak{SL}(N)$ *(*$l \in \{1, 2\}$*) satisfy* $p_i \parallel q_i$ *(*$i \in \{1, 2\}$*), then* $p_1 \perp_\alpha^{wk} p_2$ *iff* $q_1 \perp_\alpha^{wk} q_2$.

Proof.

1. Let $\mathcal{N} := \{N_i : i < \delta\}$ be a resolution of N which witnesses that N is a limit model and $f(p), f(q) \downarrow_{N_\alpha}^{\mathcal{N}_\alpha} N$. Let $(N, \mathcal{N}, N', b, \alpha)$ be such that $b \models f(q)$. Notice that $f^{-1}[\mathcal{N}] := \{f^{-1}[N_i] : i < \delta\}$ is a resolution of M which witnesses that M is a limit model such that $(M, f^{-1}[\mathcal{N}], f^{-1}[N'], f^{-1}(b), \alpha)$ holds (by invariance) and $f^{-1}(b) \models q$; since $p \perp_\alpha^{wk} q$ then $p \downarrow_{f^{-1}[N_\alpha]}^{f^{-1}[\mathcal{N}_\alpha]} f^{-1}[N']$. By invariance once more, $f(p) \downarrow_{N_\alpha}^{\mathcal{N}_\alpha} N'$, so $f(p) \perp_\alpha^{wk} f(q)$. Converse follows from a similar argument as above.

2. Since M and N are limit models over $M_{\alpha+1}$, by corollary 2.2.25 (uniqueness of limit models) there exists $f : M \approx_{M_{\alpha+1}} N$. Suppose $p \perp_\alpha^{wk} q$. Notice that $p \upharpoonright M_{\alpha+1} = f(p \upharpoonright M_{\alpha+1}) \subset f(p \upharpoonright M)$ and $q \upharpoonright M_{\alpha+1} = f(q \upharpoonright M_{\alpha+1}) \subset f(q \upharpoonright M)$. Since $p \upharpoonright M_{\alpha+1} \downarrow_{M_\alpha}^{\mathcal{M}_\alpha} M_{\alpha+1}$ (by monotonicity, since $p \downarrow_{M_\alpha}^{\mathcal{M}_\alpha} N$) and $f(p \upharpoonright M) \supset p \upharpoonright M_{\alpha+1}$ satisfies $f(p \upharpoonright M) \downarrow_{M_\alpha}^{\mathcal{M}_\alpha} N$ (by invariance of s-independence, since $p \upharpoonright M \downarrow_{M_\alpha}^{\mathcal{M}_\alpha} M$) and we also have $p \downarrow_{M_\alpha}^{\mathcal{M}_\alpha} N$, then by stationarity of $p \upharpoonright M_{\alpha+1}$ (proposition 1.4.11, notice that $M_{\alpha+1}$ is universal over M_α) we have that $f(p \upharpoonright M) = p$. In a similar way we can prove $f(q \upharpoonright M) = q$. By proposition 3.3.6 (1) we have $p \upharpoonright M \perp_\alpha^{wk} q \upharpoonright M$ iff $p \perp_\alpha^{wk} q$.

3. Notice that $p_i = q_i \upharpoonright M$. So, this holds by proposition 3.3.6 (2).

$\square_{\text{Prop. 3.3.6}}$

CHAPTER 4

A stability transfer theorem in d-tame metric abstract elementary classes

Discrete *tame* AECs are a very special kind of AECs which have a categoricity transfer theorem (see [GV06a]) and a nice stability transfer theorem (see [BKV06]). In fact -under $\aleph_0$-tameness and $\aleph_0$-locality (assuming $LS(\mathcal{K}) = \aleph_0$)-, J. Baldwin, D. Kueker and M. VanDieren proved that $\aleph_0$-Galois-stability implies κ-Galois-stability for every cardinality κ. First, they prove that $\aleph_0$-Galois-stability implies $\aleph_n$-Galois stability for every $n < \omega$. An analogous argument works for getting κ-Galois-stability if $cf(\kappa) > \omega$. They proved $\aleph_\omega$-Galois-stable using ω-locality. For getting κ-Galois stability if $cf(\kappa) = \omega$ they used an analogous argument via ω-locality as well.

This chapter is devoted to proving a generalization of that theorem for d-tame metric abstract elementary classes. A corollary of this stability transfer theorem -roughly speaking- says that under d-tameness, $\aleph_0$ and $\aleph_1$-d-stability and some suitable superstablity-like assumptions -via tame independence- we have κ-d-stability for all cardinality κ.

We have to point out that our proof does not involve ω-locality as in [BKV06], and we use superstability-like assumptions on tame-independence to get our results.

We refer to the results about independence and d-tameness given in section 1.5.

First, we provide a general stability transfer theorem.

Theorem 4.0.7 (stability transfer theorem). *Let $\mathcal{K}$ be a μ-**d**-tame (for some $\mu < \kappa$) MAEC. Suppose that for every $\theta < \kappa$ there exists $\theta \leq \theta' < \kappa$ such that $\mathcal{K}$ is θ'-**d**-stable. Define*

$$\lambda := \min\{\theta < \kappa : \mu < \theta \textit{ and } \mathcal{K} \textit{ is } \theta\text{-}\mathbf{d}\text{-stable}\}$$

,

$$\zeta := \min\{\xi : 2^\xi > \lambda\}$$

and

$$\zeta^* := \max\{\mu^+, \zeta\}.$$

If $cf(\kappa) \geq \zeta^*$ *and* $\kappa > \zeta^*$ *then* $\mathcal{K}$ *is* κ-$\mathbf{d}$*-stable.*

Proof. Suppose that this proposition is false. Let $M \in \mathcal{K}$ be a model of density character κ such that there are a_i ($i < \kappa^+$) such that $\mathbf{d}(\text{ga-tp}(a_i/M), \text{ga-tp}(a_j)/M) \geq \varepsilon$ for every $i < j < \kappa^+$ and for some fixed $\varepsilon > 0$.

Without loss of generality, we may assume that M is the completion of the union of a $\prec_{\mathcal{K}}$-increasing sequence $(M_i : i < cf(\kappa))$ such that $LS(\mathcal{K}) \leq dc(M_i) < \kappa$ and M_{i+1} is universal over M_i for every $i < cf(\kappa)$; this is possible by Fact 2.1.1 and cofinal-d-stability: By $[LS(\mathcal{K}), \kappa)$-cofinal-d-stability, let $LS(\mathcal{K}) \leq \kappa_0 < \kappa$ be a cardinality such that $\mathcal{K}$ is κ_0-d-stable and $M_0 \prec_K M$ be a model of density character κ_0. Since $\mathcal{K}$ is $dc(M_0)$-d-stable, we may construct an universal model M_1' over M_0 of density character $dc(M_0) < \kappa$ (by Fact 2.1.1). Let $B := \{b_i : i < \kappa\}$ a dense subset of M of size κ. Let $M_1 \supset M_1' \cup \{b_0\}$ be of density character $dc(M_1')$. Let $M_\alpha' := \overline{\bigcup_{i<\alpha} M_i}$ where $\alpha < \kappa$ is a limit ordinal. Notice that M_α' has still density character less than κ. By $[LS(\mathcal{K}), \kappa)$-cofinal-d-stability, there exists $dc(M_\alpha) \leq \kappa_\alpha < \kappa$ such that $\mathcal{K}$ is κ_α-d-stable. Let $M_\alpha' \prec_{\mathcal{K}} M_\alpha \prec_{\mathcal{K}} M$ be of density character κ_α. By Fact 2.1.1, let $M_{\alpha+1}$ be a universal model over M_α which contains b_α of density character $dc(M_\alpha) < \kappa$. Notice that $M' := \overline{\bigcup_{i<\kappa} M_i} \succ_K M$ and $dc(M') \leq \kappa$. Notice that since $dc(ga - S(M)) \geq \kappa^+$, then $dc(ga - S(M')) \geq \kappa^+$: in fact, if $p, q \in ga - S(M')$ then $d(p, q) := \inf\{d(a, b) : a \models p, b \models q\} \geq d(p \restriction M, p \restriction M) := \inf\{d(a, b) : a \models p \restriction M, b \models q \restriction M\}$ because any pair of realizations of p and q are realizations of $p \restriction M$ and $q \restriction M$ respectively. Let us consider the following two cases: If $dc(M') < \kappa$, this contradicts the cofinal d-stability below κ; if $dc(M') = \kappa$, consider M' instead of M.

By proposition 1.5.9, for every $\varepsilon > 0$ and every $i < \kappa^+$ there exists $M_{i,\varepsilon} \prec_{\mathcal{K}} M$ of density character $< \zeta^*$ such that $a_i \mathop{\smile\!\!\!\!\!\!|}\,^{T,\varepsilon}_{M_{i,\varepsilon}} M$. Since $dc(M_{i,\varepsilon}) < \zeta^* \leq cf(\kappa)$, there exists $j_i < cf(\kappa)$ such that $M_{i,\varepsilon} \prec_{\mathcal{K}} M_{j_i}$. By monotonicity of $\mathop{\smile\!\!\!\!\!\!|}\,^{T,\varepsilon}$ -proposition 1.5.7-, we have that $a_i \mathop{\smile\!\!\!\!\!\!|}\,^{T,\varepsilon}_{M_{j_i}} M$. By pigeon-hole principle, there exists $i^* < cf(\kappa)$ and $X \subset \kappa^+$ of size κ^+ such that for every $k \in X$ we have that $a_k \mathop{\smile\!\!\!\!\!\!|}\,^{T,\varepsilon}_{M_{j_{i^*}}} M$. By proposition 1.5.12, there exists $\delta > 0$ such that $\mathbf{d}(\text{ga-tp}(a_k/M_{j_{i^*}+1}), \text{ga-tp}(a_j/M_{j_{i^*}+1})) \geq \delta$. Since by hypothesis there exists $dc(M_{j_{i^*}+1}) \leq \theta' < \kappa$ such that $\mathcal{K}$ is θ'-$\mathbf{d}$-stable, we can take $M^* \succ_{\mathcal{K}} M_{j_{i^*}+1}$ with density character θ', so $\mathbf{d}(\text{ga-tp}(a_k/M^*), \text{ga-tp}(a_j/M^*)) \geq \delta$ for every $j \neq k \in X$ (contradicts θ'-$\mathbf{d}$-stability). $\square_{\text{Prop. 4.0.7}}$

In Theorem 4.0.7 we require that $\kappa > \zeta^*$: Notice that $\zeta^* := \max\{\mu^+, \zeta\}$, then $\lambda \geq \zeta := \min\{\xi : 2^\xi > \lambda\}$ and $\lambda := \min\{\theta : \mu < \theta < \kappa \text{ and } \mathcal{K} \text{ is } \theta\text{-d-stable}\} \geq \mu^+$. Therefore $\lambda \geq \zeta^* := \max\{\mu^+, \zeta\}$. If $\kappa = \zeta^*$, then we get a contradiction because $\kappa = \zeta^* \leq \lambda < \kappa$.

The following corollary lets us go up from d-stability in $\aleph_0$ and $\aleph_1$ to d-stability in $\aleph_n$ for every $n < \omega$.

Corollary 4.0.8. *Let $\mathcal{K}$ be an $\aleph_0$-**d**-tame MAEC. Suppose that $\mathcal{K}$ is $\aleph_0$-d-stable and $\aleph_1$-d-stable. Then $\mathcal{K}$ is $\aleph_n$-d-stable for all $n < \omega$*

Proof. Consider $\mu := \aleph_0$ and $\kappa := \aleph_2$. Notice that $\lambda := \min\{\theta < \kappa : \mu < \theta$ and $\mathcal{K}$ is θ-**d**-stable $\} = \aleph_1$ and $\zeta := \min\{\xi : 2^{\xi} > \lambda\} \leq \aleph_1$. So, $\zeta^* := \max\{\mu^+, \zeta\} = \aleph_1$ (independently if CH holds). In this case, $a \mathop{\smile\!\!\!\!\!\mid}^{T}_{N} M$ means that given ε there exists a separable model $N_\varepsilon \prec_{\mathcal{K}} N$ such that $a \mathop{\smile\!\!\!\!\!\mid}^{T,\varepsilon}_{N_\varepsilon} M$. Notice that $cf(\kappa) = \aleph_2 \geq \zeta^* = \aleph_1$, so by theorem 4.0.7 we have that $\mathcal{K}$ is $\aleph_2$-d-stable. By an inductive argument, we have that $\mathcal{K}$ is $\aleph_n$-d-stable for all $n < \omega$. $\square_{\text{Cor. 4.0.8}}$

The following corollary says that, under the superstability-like assumption below, we can get $\aleph_\omega$-d-stability from d-stability in $\aleph_n$ for every $n < \omega$.

Assumption 4.0.9 (ε-local character). *For every tuple $\overline{a}$, every $\varepsilon > 0$, every infinite ordinal σ and every increasing and continuous $\prec_{\mathcal{K}}$-chain of models $\langle M_i : i < \sigma \rangle$, there exists $j < \sigma$ such that $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{T,\varepsilon}_{M_j} \overline{\bigcup_{i<\sigma} M_i}$.*

Corollary 4.0.10. *Let $\mathcal{K}$ be a $\aleph_0$-d-tame, $\aleph_0$-d-stable and $\aleph_1$-d-stable MAEC which satisfies assumption 4.0.9. Then $\mathcal{K}$ is $\aleph_\omega$-d-stable.*

Proof. By corollary 4.0.8, $\mathcal{K}$ is $\aleph_n$-d-stable for all $n < \omega$. By reductio ad absurdum, suppose $\mathcal{M}$ is not $\aleph_\omega$-d-stable. So, there exists $M \in \mathcal{K}$ of density character $\aleph_\omega$ such that $dc(\text{ga-S}(M)) \geq \aleph_{\omega+1}$. Without loss of generality, we may assume M is the completion of the union of a $\prec_{\mathcal{K}}$-increasing and continuous chain $\{M_n : i < \omega\}$ where $dc(M_n) = \aleph_n$ and M_{n+1} is universal over M_n for all $n < \omega$. So, there exist $\varepsilon > 0$ and $a_i \in \mathbb{M}$ ($i < \aleph_{\omega+1}$) such that $d(\text{ga-tp}(a_i/M), \text{ga-tp}(a_j/M)) \geq \varepsilon$ for all $i \neq j < \aleph_{\omega+1}$ (we can find them using the same argument when the space is not separable, because $cf(\aleph_{\omega+1}) > \omega$, see [Lim93, Wil70]).

By $\aleph_0$-d-tameness, there exists $\delta_\varepsilon > 0$ such that for every $p, q \in \text{ga-S}(M)$, if $d(p,q) \geq \varepsilon$ then there exists $M' \prec_{\mathcal{K}} M$ of density character $\aleph_0$ such that $d(p \restriction M', q \restriction M') \geq \delta_\varepsilon$ (see definition 1.5.2). Define $\delta := \delta_\varepsilon/3$.

On the other hand, given $i < \aleph_{\omega+1}$, by the superstability-like assumption 4.0.9 there exists $n_i < \omega$ such that $a_i \mathop{\smile\!\!\!\!\!\mid}^{T,\delta}_{M_{n_i}} M$. Since $cf(\aleph_{\omega+1}) = \aleph_{\omega+1} > \omega$, by pigeon-hole principle there exists a fixed $n < \omega$ and $X \subset \aleph_{\omega+1}$ of size $\aleph_{\omega+1}$ such that $a_i \mathop{\smile\!\!\!\!\!\mid}^{T,\delta}_{M_n} M$ for all $i \in X$.

Notice that for every $i \neq j \in X$, $d(\text{ga-tp}(a_i/M), \text{ga-tp}(a_j/M)) \geq \varepsilon$ and $a_i, a_j \mathop{\smile\!\!\!\!\!\mid}^{T,\delta}_{M_n} M$. We may say that

$$d(\text{ga-tp}(a_i/M_{n+1}), \text{ga-tp}(a_j/M_{n+1})) \geq \delta.$$

If not, suppose $d(\text{ga-tp}(a_i/M_{n+1}), \text{ga-tp}(a_j/M_{n+1})) < \delta$. Let $N^* \prec_{\mathcal{K}} M_n$ be a model of size $\aleph_0$ which witnesses $a_i, a_j \downarrow^{T,\delta}_{M_n} M$. Let $M^\circ \prec_{\mathcal{K}} M$ be any model of density character $\aleph_0$. Let $M^* \prec_{\mathcal{K}} M$ be a model of density character $\aleph_0$ which contains $|N^*| \cup |M^\circ|$. Since M_{n+1} is universal over M_n, so it is universal over N^*. Therefore, there exist a model M' such that $N^* \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} M_{n+1}$ and an isomorphism $f : M' \overset{f}{\cong}_{N^*} M^*$. Since N^* witnesses that $a_i, a_j \downarrow^{T,\delta}_{M_n} M$ and $N^* \prec_{\mathcal{K}} M' \overset{f}{\cong}_{N^*} M^* \prec_{\mathcal{K}} M$, therefore

$$\mathbf{d}(\text{ga-tp}(a_i/M^*), \text{ga-tp}(f(a_i)/M^*)) < \delta$$

and

$$\mathbf{d}(\text{ga-tp}(a_j/M^*), \text{ga-tp}(f(a_j)/M^*)) < \delta$$

Since $M' \prec_{\mathcal{K}} M_{n+1}$, we have that

$$\begin{aligned}\mathbf{d}(\text{ga-tp}(a_i/M'), \text{ga-tp}(a_j/M')) &\leq \mathbf{d}(\text{ga-tp}(a_i/M_{n+1}), \text{ga-tp}(a_j/M_{n+1})) \\ &< \delta\end{aligned}$$

so,

$$\begin{aligned}\mathbf{d}(\text{ga-tp}(f(a_i)/M^*), \text{ga-tp}(f(a_j)/M^*)) &= \mathbf{d}(\text{ga-tp}(a_i/M'), \text{ga-tp}(a_j/M')) \\ &< \delta\end{aligned}$$

Therefore:

$$\begin{aligned}\mathbf{d}(\text{ga-tp}(a_i/M^\circ), \text{ga-tp}(a_j/M^\circ)) &\leq \mathbf{d}(\text{ga-tp}(a_i/M^*), \text{ga-tp}(a_j/M^*)) \\ &\leq \mathbf{d}(\text{ga-tp}(a_i/M^*), \text{ga-tp}(f(a_i)/M^*)) \\ &\quad +\mathbf{d}(\text{ga-tp}(f(a_i)/M^*), \text{ga-tp}(f(a_j)/M^*)) \\ &\quad +\mathbf{d}(\text{ga-tp}(f(a_j)/M^*), \text{ga-tp}(a_j/M^*)) \\ &< 3\delta = \delta_\varepsilon\end{aligned}$$

Notice that M° is an arbitrary $\mathcal{K}$-submodel of M of density character $\aleph_0$. Therefore, by $\aleph_0$-d-tameness, we have that $\mathbf{d}(\text{ga-tp}(a_i/M), \text{ga-tp}(a_j/M)) < \varepsilon$ (contradiction).

Hence $dc(\text{ga-S}(M_{n+1})) \geq \aleph_{\omega+1} > \aleph_{n+1}$, contradicting $\aleph_{n+1}$-d-stability. $\square_{\text{Cor. 4.0.10}}$

Corollary 4.0.11 (weak superstability). *Let $\mathcal{K}$ be an $\aleph_0$-d-tame, $\aleph_0$-d-stable and $\aleph_1$-d-stable MAEC, which also satisfies assumption 4.0.9 (countable character of ε-splitting). Then $\mathcal{K}$ is κ-d-stable for every cardinality κ.*

Proof. By induction on all cardinalities $\kappa \geq \aleph_0$, we prove that $\mathcal{K}$ is κ-d-stable. By hypothesis, we have $\mathcal{K}$ is $\aleph_0$ and $\aleph_1$-d-stable.

Suppose $\mathcal{K}$ is λ-d-stable for all $\lambda < \kappa$. Notice that $\mu = \aleph_0$, $\lambda = \min\{\theta > \mu : \mathcal{K} \text{ is } \theta\text{-d-stable}\} = \aleph_1$, $\zeta = \min\{\xi : 2^\xi > \lambda\} \leq \aleph_1$ and $\zeta^* = \max\{\mu^+, \zeta\} = \aleph_1$. Notice that we have to check κ-d-stability for every $\kappa \geq \aleph_2 > \aleph_1 = \zeta^*$. If $cf(\kappa) > \aleph_0$ then $cf(\kappa) \geq \aleph_1 = \zeta^*$, then by theorem 4.0.7 $\mathcal{K}$ is κ-d-stable.

If $cf(\kappa) = \omega$, the argument given in corollary 4.0.10 works for proving that $\mathcal{K}$ is κ-d-stable. For the sake of completeness, we provide the proof if $cf(\kappa) = \omega$. Let $\Lambda : \aleph_0 \to \kappa$ be a cofinal mapping. By hypothesis, $\mathcal{K}$ is $\Lambda(n)$-d-stable. By reductio ad absurdum, suppose $\mathcal{M}$ is not κ-d-stable. So, there exists $M \in \mathcal{K}$ of density character κ such that $dc(\text{ga-S}(M)) \geq \kappa^+$. *Without loss of generality*, we may assume M is the completion of the union of a $\prec_{\mathcal{K}}$-increasing and continuous chain $\{M_n : i < \omega\}$ where $dc(M_n) = \Lambda(n)$ and M_{n+1} is universal over M_n for all $n < \omega$. Given $\varepsilon > 0$, let $a_i \in \mathbb{M}$ $(i < \kappa^+)$ be such that $d(\text{ga-tp}(a_i/M), \text{ga-tp}(a_j/M)) \geq \varepsilon$ for all $i \neq j < \kappa^+$. Let $\delta := \delta_\varepsilon/3$ (where δ_ε is given in definition 1.5.2 -tameness-). On the other hand, given $i < \kappa^+$, by the superstability-like assumption 4.0.9 there exists $n_i < \omega$ such that $a_i \downarrow^{T,\delta}_{M_{n_i}} M$. Since $cf(\kappa^+) = \kappa^+ > \omega$, by the pigeon-hole principle there exists a fixed $n < \omega$ and $X \subset \kappa^+$ of size κ^+ such that $a_i \downarrow^{T,\delta}_{M_n} M$ for all $i \in X$.

Notice that for every $i \neq j \in X$, $d(\text{ga-tp}(a_i/M), \text{ga-tp}(a_j/M)) \geq \varepsilon$ and $a_i, a_j \downarrow^{T,\delta}_{M_n} M$. So, by the argument given in corollary 4.0.10 we may say

$$d(\text{ga-tp}(a_i/M_{n+1}), \text{ga-tp}(a_j/M_{n+1})) \geq \delta.$$

Hence $dc(\text{ga-S}(M_{n+1})) \geq \kappa^+ > \Lambda(n+1)$, which contradicts $\Lambda(n+1)$-d-stability. $\square_{\text{Cor. 4.0.11}}$

CHAPTER 5

Exploring some examples: Bounded and unbounded operators

In [SS78], S. Shelah and J. Stern proved that the Hanf number of classes of metric structures is far away from the behavior of first order elementary classes. Despite they did not calculate the exact Hanf number for this kind of classes -they just said that this is exactly the Hanf number of the second order theory of binary relations-, we know that this is a large cardinality because it is related to a language that allows quantification over countable sets. This implies in particular that classical first order model theoretical techniques do not apply directly to those classes.

Additionally, in [She75] Shelah proposes the study of the model theory of analytical structures. Around 1974, there were few responses to this challenge, but the work of Henson -in the late 90's- and later Henson and Iovino -[HI02]- in 2002 opened the way to the development of a genuine first order continuous approach to the model theory of metric structures.

Ben Yaacov used a new approach to study metric structures: Compact Abstract Theories -shortly, CATs, see [BY03]-. In this approach, Ben Yaacov studied metric structure classes as Positive Robinson Theories and as Compact Elementary Categories.

Lately, in 2008 Ben Yaacov, Berenstein, Henson and Usvyatsov presented in a monography -see [BYBHU08]- a new approach to study metric structures based on [CK66], where the underlying logic is constructed using uniformly continuous functions on finite powers of $[0, 1]$, instead of a compact space X as in [CK66]. All of those approaches are equivalent, but we based our definitions in the Ben Yaacov-Berenstein-Henson-Usvyatsov's setting, as in [Hir06, HH09]

Metric Abstract Elementary Classes (for short, MAECs) are devoted to provide a general framework to study metric classes which are not axiomatizable in CL, although elementary CL classes are also particular examples of MAECs. C. Argoty gave an example of a non-axiomatizable in CL superstable MAEC (*Hilbert Spaces with an unbounded closed self-adjoint operator*, [Arg1X]). In his example, Argoty gave an interesting characterization of Galois types via measures using the Spectral Theory of this kind of operators and studied notions of almost orthogonality -via orthogonality of measures- and domination of types over the empty set -via absolute continuity of measures-.

Å. Hirvonen and T. Hyttinen studied a more general framework to study examples which are not ω-d-stable -in the sense defined in chapter 1- but ω-d-stable up to perturbations (see [HH11]).

This chapter is devoted to study some of these examples and to interpret the notions studied in chapter 3.

On the other hand, *Gelfand triplets* (also known as *Rigged Hilbert Spaces*) are very important in Quantum Mechanics because they provide a natural mathematical setting to study the Dirac's bra-ket formalism -which cannot be completely studied just with Hilbert Spaces- [Mad]. In this chapter, we begin the study of Gelfand triplets as MAECs.

5.1 Hilbert spaces with a unitary operator.

In this section, we study some consequences of corollary 3.1.10 in the class of Hilbert spaces expanded to a unitary operator (example given by C. Argoty and A. Berenstein, see [AB09]).

Notation 5.1.1. Let $\mathcal{K}_u := \{(\mathcal{H}, U) \mid \mathcal{H}$ is a Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator$\}$. We denote by $\langle\ ,\ \rangle_{\mathcal{H}}$ the inner product defined on $\mathcal{H}$. We drop $\mathcal{H}$ if it is clear the space on which the inner product is defined.

Fact 5.1.2 (Argoty-Berenstein, see [AB09]). *$\mathcal{K}_u$ is an axiomatizable class in (first order) Continuous Logic.*

Fact 5.1.3 (Argoty-Berenstein, see [AB09]). *$\mathcal{K}_u$ satisfies Amalgamation Property.*

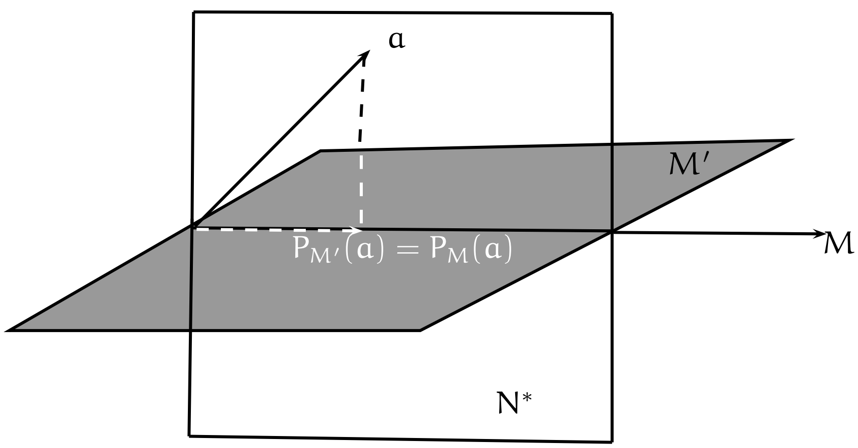
Fact 5.1.4 (Argoty-Berenstein, see [AB09]). *$\mathcal{K}_u$ is ω-d-stable.*

Definition 5.1.5. Let $(\mathcal{H}, U)$ be a monster model in $\mathcal{K}_u$, $\overline{a} \in \mathcal{H}$ and $C, B \subset \mathcal{H}$ be small sets (i.e.: $dc(B), dc(C) < dc(\mathcal{H})$). Let σ be the spectrum of U and P_λ the respective projection associated to $\lambda \in \sigma$. We denote by $\overline{a} \downarrow^*_C B$ the assertion that $P_{acl(B)}(P_\lambda(\overline{a})) = P_{acl(BC)}(P_\lambda(\overline{a}))$ for every $\lambda \in \sigma$.

Fact 5.1.6 (Argoty-Berenstein, see [AB09]). $\overline{a} \downarrow^*_C B$ *if and only if* $P_{acl(C)}(\overline{a}) = P_{acl(BC)}(\overline{a})$.

Fact 5.1.7 (Argoty-Berenstein, see [AB09]). $\downarrow^*$ *corresponds to an independence notion in continuous logic, so it is equivalent to non-forking independence.*

Remark 5.1.8. Corollary 3.1.10 says in the example above that given $\overline{a} \in \mathcal{H}$ and $M \in \mathcal{K}_u$ such that M is saturated enough and $\overline{a} \notin M$, there exists $N^* \supset acl(M\overline{a})$ in $\mathcal{K}_u$ such that for every $M' \supset M$ in $\mathcal{K}_u$, $P_M(\overline{a}) = P_{M'}(\overline{a})$ implies that $P_M(b) = P_{M'}(b)$ for every $b \in N^*$; i.e.: $\overline{a}$ determines the projections on M of all elements in N^*.



Remark 5.1.9. In Hilbert spaces together with a unitary operator, $p \perp^{wk} q$ (both of them types over a Hilbert space $M \in \mathcal{K}_u$) means that for every Hilbert space $N \geq M$ in $\mathcal{K}_u$which contains a realization of q and for any fixed realization a of p, $P_M(a) = P_N(a)$. Let us consider $M = \langle 0 \rangle$ (despite $M \notin \mathcal{K}_u$) and $N = \langle b \rangle$, notice that weak orthogonality implies that $0 = P_M(a) = P_N(a)$. Therefore, $p \perp^{wk} q$ implies any realizations of p and q respectively are orthogonal in the sense of the inner product in Hilbert spaces.

5.2 L_p spaces

In this section, we will study an example due to I. Ben-Yaacov, A. Berenstein and C.W. Henson about the class of L^p spaces on decomposable measures (see [BBH09]).
Let X be a set, U a σ-algebra on X and μ a measure on U, and let $p \in [1, \infty)$. Consider $\mathcal{K}_p$ the completion of models of $(Th_{\mathcal{A}}(L^p(X, U, \mu))$ in the setting of *Positive Bounded Theories* (see [HI02]). Throughout this section, we just consider decomposable measures.

Definition 5.2.1. Let E be an abstract L^p space. We say that E is a *Banach lattice* if $\|x + y\|^p = \|x\|^p + \|y\|^p$ whenever $x, y \in E$ and $x \wedge y = 0$.

Example 5.2.2. $L^p(X, U, \mu)$ is a Banach lattice, defining $\wedge$ pointwise.

Definition 5.2.3. Given E a Banach lattice and $f \in E$, we define:

1. $f^+ := f \vee 0$ (which is called *positive part* of f).
2. $f^- := (-f)^+$ (which is called *negative part* of f)
3. $|f| := f^+ + f^-$.

f is said to be *positive* if and only if $f = f^+$ and *negative* if and only if $-f$ is positive.

Definition 5.2.4. Let $B \subset E$ where E is a Banach lattice. $B^\perp := \{f \in E : |f| \wedge |g| = 0 \text{ for all } g \in B\}$

Fact 5.2.5. *Given a Banach lattice* E*, there exists a (decomposable) measure space* (X, U, μ) *such that* $E \cong L^p(X, U, \mu)$.

Fact 5.2.6 (Conditional expectation). *Let* $(Y, V, \mu) \subset (X, U, \mu)$ *be measure spaces,* $1 \le p < \omega$ *and* $f \in L^p(X, U, \mu)$*. Then there exists a unique* $g_f \in L^p(Y, V, \mu)$ *such that* $\int_A g_f d\mu = \int_A f d\mu$ *for every* $A \in V$*. We call* g_f *the conditional expectation of* f *with respect to* (Y, V, μ) *and we denote it by* $\mathbb{E}(f|V)$*. The operator mapping* $f \mapsto \mathbb{E}(f|V)$ *is a contractive, positive projection from* $L^p(X, U, \mu)$ *onto* $L^p(Y, V, \mu)$ *and* $\mathbb{E}(f|V) = 0$ *for any* $f \in L^p(Y, V, \mu)^\perp$.

Definition 5.2.7. Let $(Y, V, \mu) \subset (X, U, \mu)$ be measure spaces. Let $\overline{f} := (f_1, \cdots, f_n)$, $\overline{g} := (g_1, \cdots, g_n) \in L^p(X, U, \mu)^n$ such that $\overline{f}, \overline{g} \in (L^p(Y, V, \mu))^{\perp\perp}$. We say that $\overline{f}$ and $\overline{g}$ have the same (joint) conditional distribution over (Y, V, μ) if and only if for any Borel set $B \subset \mathbb{R}^n$ we have $\mathbb{E}(\overline{f}^{-1}(B)|V) = \mathbb{E}(\overline{g}^{-1}(B)|V)$, and we denote this fact by $\mathrm{dist}(\overline{f}|V) = \mathrm{dist}(\overline{g}|V)$.

Fact 5.2.8. *Let* $\mathcal{U}$ *be an abstract* L^p *Banach laticce and let* C *be any sublattice of* $\mathcal{U}$*. There exists a unique linear operator* $T : \mathcal{U} \to C$ *such that* T *is an contractive, positive projection and* $T(f) = 0$ *for any* $f \in C^\perp$.

Notation 5.2.9. Let $C = L^p(Y, V, \mu) \subset \mathcal{U} = L^p(X, U, \mu)$ be Banach lattices. The unique contractive, positive projection from $\mathcal{U}$ onto C that is identically 0 on $C^\perp$ will be denoted by $\mathbb{E}_C^{\mathcal{U}}$. We drop the superscript $\mathcal{U}$ if it is not ambiguous.

Definition 5.2.10. Let $(X, V, \mu) \subset (X, U, \mu)$. The measure space (X, U, μ) is said to be *atomless* over (X, V, μ) if and only if for every $A \in U$ of positive measure there exists $B \in U$ such that $A \cap B \neq A \cap C$ for all $C \in V$

Fact 5.2.11. *Let* $(X, V, \mu) \subset (X, U, \mu)$ *be measure spaces, where* (X, U, μ) *is atomless over* (X, V, μ)*. Let* $(X, V, \mu) \subset (X, W, \mu)$ *be any other extension. Then for any* $\overline{f} \in L^p(X, W, \mu)^n$*, there is* $\overline{g} \in L^p(X, U, \mu)^n$ *such that* $\mathrm{dist}(\overline{f}|V) = \mathrm{dist}(\overline{g}|V)$.

Fact 5.2.12. *Let* B *be a sublattice of* $\mathbb{M}$ *and let* $(Y, V, \mu) \subset (X, U, \mu)$ *be measure spaces such that* $B = L^p(Y, V, \mu)$ *and* $\mathbb{M} = L^p(X, U, \mu)$*. Let* $\overline{f} \in \mathbb{M}^n$ *and* $\overline{h} \in \mathbb{M}^n$ *be such that* $\overline{f}, \overline{h} \in (B^{\perp\perp})^n$*. Then* $\mathrm{dist}(\overline{f}|V) = \mathrm{dist}(\overline{g}|V)$ *if and only if* $\mathrm{tp}(\overline{f}/B) = \mathrm{tp}(\overline{h}/B)$.

Definition 5.2.13. Let $A, B, C \le \mathcal{U}$ be sublattices of $\mathcal{U}$ such that $C \le A \cap B$. Let $\mathbb{E}_B$ and $\mathbb{E}_C$ be the conditional expectation projections to B and C respectively, given by fact 5.2.8. We say that A is $*$-independent from Bover C (denoted by $A \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$) if and only if $\mathbb{E}_B(f) = \mathbb{E}_C(f)$ for every $f \in A$.

Fact 5.2.14. *The theory of* L^p *Banach Lattices is stable, and non-dividing (see [BYBHU08]) coincides with* $*$*-independence.*

5.2.1 What the results in the chapter 3 say in this example

Recall the claim of corollary 3.1.10: Given $(M, \mathcal{M}, \bar{f}, N)$ such that $\bar{f} \downarrow_{M_0}^{\mathcal{M}_0} M$ (and therefore ga-tp$(\bar{f}/M)$ is a stationary type because M is a universal model over M_0), there exist N^* and a resolution $\mathcal{M}^*$ which witnesses that M is a limit model over M_0 such that $\bar{f} \bowtie_M^{\mathcal{M}^*} N^*$.

This says that if $M = L^p(X, U, \mu)$ and $M_0 = L^p(X, U_0, \mu) \leq M$ are such that (X, U, μ) is atomless over (X, U_0, μ), we can find a Banach Lattice $N^* = L^p(Y, V, \mu) \geq M = L^p(X, U, \mu)$ such that $\mathbb{E}_{M'}(\bar{f}) = \mathbb{E}_M(\bar{f})$ if and only if $\mathbb{E}_{M'}(\bar{g}) = \mathbb{E}_M(\bar{g})$ for all $\bar{g} \in N^*$.

In this setting, weak orthogonality ($p \perp^{wk} q$ over M) means that given a Banach lattice $N \succ_{\mathcal{K}} M$ which contains a realization of q, there exists an extension ga-tp$(f/N) \supset p$ such that $\mathbb{E}_N(f) = \mathbb{E}_M(f)$.

In this example, for $p \in$ ga-S(M_1) and $q \in$ ga-S(M_2), $p \parallel q$ means that for every $M \supset M_1 \cup M_2$, if $p', q' \in$ ga-S(M) are the independent extensions of p and q respectively (i.e.: $\mathbb{E}_{M_1^0}(f) = \mathbb{E}_M(f)$ and $\mathbb{E}_{M_2^0}(g) = \mathbb{E}_M(g)$ for some $f \models p'$ and $g \models q'$), then $p' = q'$; i.e.: p and q are parallel if and only if equality between the conditional expectations $\mathbb{E}_M$, $\mathbb{E}_{M_1^0}$ and $\mathbb{E}_{M_2^0}$ determines equality between the independent extensions of p and q over M.

Proposition 3.3.6 (3) says that weak orthogonality is preserved under paralellism; i.e.: if $p_i \in$ ga-S(M_1) and $q_i \in$ ga-S(M_2) ($i \in \{1, 2\}$) are such that $p_i \parallel q_i$ and $p_1 \perp^{wk} p_2$, then $q_1 \perp^{wk} q_2$. It means that if equality between the conditional expectations $\mathbb{E}_M$, $\mathbb{E}_{M_1^0}$ and $\mathbb{E}_{M_2^0}$ determines equality between the independent extensions of p_i and q_i over $M \supset M_1 \cup M_2$, then this implies equality between $\mathbb{E}_N$ and $\mathbb{E}_{M_1}$ (where N is a model which contains a realization of p_2) is equivalent to have equality between $\mathbb{E}_{N'}$ and E_{M_2} (where N is a model which contains a realization of q_2).

5.3 Hilbert spaces with an unbounded closed self-adjoint operator

This section is devoted to study the class of Hilbert spaces together with an unbounded closed self-adjoint operator. The beginning of the model-theoretical analysis of this example is due to C. Argoty (see [Arg1X]). This example is interesting because it is a non-axiomatizable in CL MAEC. First, we give some basic definitions related to this example. In the section 5.3.1, we give some basic facts about independence, domination and orthogonality in this example.

This example is constructed in the following way: Let H be a complex Hilbert space with a unbounded closed selfadjoint operator Q. Consider the following structure: $(H, 0, -, i, x+y, \|\cdot\|, \Gamma_Q)$ where 0 is the zero vector in H; $- : H \to H$ is defined by $v \mapsto -v$; $i : H \to H$ is defined by $v \mapsto iv$ (where $i^2 = -1$; $x+y : X \times H \to H$ is defined by $(x, y) \mapsto x+y$, $\|\cdot\| : H \to R^+ \cup \{0\}$ is the norm of H; $\Gamma_Q : H \times H \to R^+ \cup \{0\}$ is defined by $\Gamma_Q(v, w) := \mathtt{dist}((x, y), \mathtt{graph}(Q))$. This structure is denoted by (H, Q).

Definition 5.3.1. Let Q_1 and Q_2 be closed unbounded selfadjoint operators defined on Hilbert spaces H_1 and H_2 respectively. We say that Q_1 and Q_2 are *essentially equivalent* (denoted by $Q_1 \sim_\sigma Q_2$) if and only if:

1. $\sigma(Q_1) = \sigma(Q_2)$
2. $\sigma_e(Q_1) = \sigma_e(Q_2)$
3. $\dim\{x \in H_1 : Q_1 x = \lambda x\} = \dim\{x \in H_2 : Q_2 x = \lambda x\}$ for every $\lambda \in \sigma(Q_1) \setminus \sigma_e(Q_1)$

Definition 5.3.2. Given $v \in H$ and $A \subset_{\mathrm{Borel}} \mathbb{R}$, define $\mu_v(A) := \langle \chi_A(Q)v \mid v \rangle$. μ_v is called the *spectral measure defined by* v.

Notation 5.3.3. $K_{(H,Q)} := \{(H', Q') : (H', Q')$ is an L-structure and $Q' \sim_\sigma Q\}$

Fact 5.3.4. $K_{(H,Q)}$ *together with* $\subset$ *(to be a subspace) is a homogeneous MAEC, with* $LS(K) \leq 2^{2^{\aleph_0}}$, *which satisfies JEP and AP.*

Reference. [Arg1X] □

Fact 5.3.5. *Let* $v_1 \in (H_1, Q_1)$ *and* $v_2 \in (H_2, Q_2)$. *Then* $\text{ga-tp}_{(H_1,Q_1)}(v_1/\emptyset) = \text{ga-tp}_{(H_2,Q_2)}(v_2/\emptyset)$ *if and only if* $\mu_{v_1} = \mu_{v_2}$.

Reference. [Arg1X] □

Fact 5.3.6. *Let* $v, w \in \mathbb{M}$ *and* $G \subset \mathbb{M}$. *Then* $\text{ga-tp}(v/G) = \text{ga-tp}(w/G)$ *if and only if* $P_G v = P_G w$ *and* $\text{ga-tp}(P_{G^\perp} v/\emptyset) = \text{ga-tp}(P_{G^\perp} w/\emptyset)$

Reference. [Arg1X] □

5.3.1 Independence, domination and orthogonality

Definition 5.3.7. Let $v \in \mathbb{M}$ and let $F, G \subset \mathbb{M}$. We say that v is independent from G over F if $P_{\mathrm{acl}(F)} v = P_{\mathrm{acl}(F \cup G)} v$ and denote it by $v \mathop{\smile\!\!\!\!\!\mid}^*_F G$.

Fact 5.3.8. $\mathcal{K}_{(H,Q)}$ *is stable and* $\mathop{\smile\!\!\!\!\!\mid}^*$ *is a freeness relation, so* $\mathop{\smile\!\!\!\!\!\mid}^*$ *corresponds to (continuous) first order non-forking.*

Reference. [Arg1X] □

Fact 5.3.9. *Let* $p, q \in S_1(\emptyset)$, *let* $v \models p$ *and* $w \models q$. *Then,* $p \perp^a q$ *if and only if* $\mu_{v_e} \perp \mu_{w_e}$.

Reference. [Arg1X] □

Fact 5.3.10. *Let* $p, q \in S_1(G)$*, let* $v \models p$ *and* $w \models q$ *. Then,* $p \perp^a q$ *if and only if* $\mu_{P_{G^\perp} v_e} \perp \mu_{P_{G^\perp} w_e}$.

Reference. [Arg1X] □

Fact 5.3.11. *Let* $p, q \in S_1(\emptyset)$*,* $v \models p$ *and* $w \models q$*. Then ,* $p \rhd_\emptyset q$ *if and only if* $\mu_{v_e} >> \mu_{w_e}$.

Reference. [Arg1X] □

Question 5.3.12. In spite of the results given by Argoty, there is no any published reference about the equivalence of domination, orthogonality and paralellism of types over non-empty sets as it was done in [Arg1X]. However, it seems that there is a similar equivalences in terms of measures as in [Arg1X]. Having such equivalences, we would have nice interpretations in this setting of the got results in chapter 3.

5.4 Gelfand triplets

We base the definitions given in this section on [Wlo73].
Let $\mathcal{H}_0 := (H_0, \langle\ ,\ \rangle_0)$ be a Hilbert space and $\mathcal{H}_+ := (H_+, \langle\ ,\ \rangle_+)$ be a Hilbert space such that $H_+ \subset_{\|\cdot\|_0-dense} H_0$ and $\|u\|_0 \leq \|u\|_+$ for every $u \in H_+$. Notice that the inclusion $i : H_+ \to H_0$ is uniformly continuous.

Fact 5.4.1. *Let* $B : \mathcal{H}_0 \times \mathcal{H}_+ \to \mathbb{C}$ *be the bi-linear form defined by* $B(u, v) := \langle u, i(v)\rangle_0$*. It is a well-known fact that there exists an operator* $I : H_0 \to H_+$ *such that* $\langle I(u), v\rangle = \langle u, i(v)\rangle = B(u, v)$ *(i.e.:* I *and* i *are "almost" adjoint).*

Reference. [Wlo73] □

Notice that $\|i\| \leq 1$, so $\|I\| \leq 1$ (since $\|u\|_0 \leq \|u\|_+$ for every $u \in H_+$).

Define $\langle u, v\rangle_- := \langle iI(u), v\rangle_0 = \langle I(u), I(v)\rangle_+$ for every $u, v \in H_0$. Notice that $\langle\ ,\ \rangle_-$ is an inner product defined on H_0. But in general, H_0 is not a complete space in the $\|\cdot\|_-$-norm. Take H_- as the $\|\cdot\|_-$-completion of H_0. Notice that $\|u\|_- \leq \|u\|_0$ for every $u \in H_0$: If $u \in \mathcal{H}_0$, notice that

$$\begin{aligned}
\|u\|_-^2 &:= \langle u, u\rangle_- \\
&= \langle iIu, u\rangle_0 \text{ (by definition)} \\
&\leq \|iIu\|_0 \cdot \|u\|_0 \text{ (by Cauchy-Schwarz inequality)} \\
&\leq \|i\| \cdot \|I\| \cdot \|u\|_0^2 \\
&\leq \|u\|_0^2 \text{ (since } \|i\| \leq 1 \text{ and } \|I\| \leq 1)
\end{aligned}$$

Definition 5.4.2 (Gelfand triplet). A triple of Hilbert spaces $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ is called a *Gelfand triplet* if

1. $\mathcal{H}_+ \subset_{\|\cdot\|_0-\text{dense}} \mathcal{H}_0 \subset_{\|\cdot\|_--\text{dense}} \mathcal{H}_-$
2. $I : \mathcal{H}_0 \to \mathcal{H}_+$ is the "almost" adjoint of the inclusion $i : \mathcal{H}_+ \to \mathcal{H}_0$,
3. $\langle u, v\rangle_- = \langle I(u), I(v)\rangle_+ = \langle i(I(u)), v\rangle_0$ for every $u, v \in H_0$
4. $\|u\|_- \leq \|u\|_0$ for every $u \in \mathcal{H}_0$
5. $\|v\|_0 \leq \|v\|_+$ for every $v \in \mathcal{H}_+$

In this section, we consider multi-sorted structures with four sorts $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-, \mathbb{C})$ with function symbols for their respective inner products and a function symbol I such that its interpretation corresponds to a linear operator $I : \mathcal{H}_0 \to \mathcal{H}_+$. Denote this kind of structures by $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-, \mathbb{C}, I)$. From now, the sort corresponding to $\mathbb{C}$ will not be explicitly displayed in our statements, but we understand that $\mathbb{C}$ is consider as a sort in any structure of the form $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-, I)$.

5.4.1 Gelfand triplet as Multisorted Metric Abstract Elementary Classes

Definition 5.4.3. Let $G := (\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-, I)$ and $G' := (\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-, I)$ be Gelfand triplets, we say that G is a substructure of G′ (which we denote by $G \leq G'$) if and only if:

1. $\mathcal{H}_\square$ is a Hilbert subspace of $\mathcal{H}'_\square$ for $\square \in \{+, 0, -\}$, and $\langle\ ,\ \rangle_\square = \langle\ ,\ \rangle'_\square \restriction H_\square$.

Fact 5.4.4. *The "almost" adjoint* I *of the inclusion in a Gelfand triplet* $G := (\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ *is interpretable in* G.

Proof. By definition of I (I is the "adjoint" of the inclusion $i : \mathcal{H}_+ \to \mathcal{H}_+$) $\square_{\text{Prop. 5.4.4}}$

Proposition 5.4.5. *Let* $G := (\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ *and* $G' := (\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-)$ *be Gelfand triplets such that* G *is a substructure of* G′ *and* I *and* I′ *are the "adjoint" operators of the inclusions in* G *and* G′ *respectively, then* $I' \restriction \mathcal{H}_0 = I$.

Proof. Let $i' : H'_+ \to H'_0$ be the inclusion in G′ and $I' : H_0 \to H_+$ be its respective "adjoint" operator. Let $u \in H_0$, we will see that $\langle I'u - Iu, I'u - Iu\rangle'_+ = 0$, so $I'u = Iu$.
Notice that $\langle I'u - Iu, I'u - Iu\rangle'_+ = \langle I'u, I'u\rangle'_+ - \langle I'u, Iu\rangle'_+ - \langle Iu, I'u'_+\rangle + \langle Iu, Iu\rangle'_+$.
Also, notice that

$$\begin{aligned}
\langle I'u, Iu\rangle'_+ &= \langle u, i'Iu\rangle'_0 \text{ (since } I' = (i')^*) \\
&= \langle u, Iu\rangle'_0 \text{ (}i'\text{ is the inclusion in } G') \\
&= \langle u, Iu\rangle_0 \text{ (since } G \leq G') \\
&= \langle u, iIu\rangle_0 \text{ (since } i \text{ is the inclusion in G)} \\
&= \langle Iu, Iu\rangle_+ \text{ (since } I = i^*) \\
&= \langle Iu, Iu\rangle'_+ \text{ (since } G \leq G')
\end{aligned}$$

and

$$\begin{aligned}\langle Iu, I'u\rangle'_+ &= \overline{\langle I'u, Iu\rangle'_0} \\ &= \overline{\langle Iu, Iu\rangle'_+} \text{ (see above)} \\ &= \langle Iu, Iu\rangle'_+\end{aligned}$$

therefore, $\langle I'u, Iu\rangle'_+ = \langle Iu, I'u\rangle'_+$. Also, we have that

$$\begin{aligned}\langle I'u, I'u\rangle'_+ &= \langle u, i'I'u\rangle'_0 \text{ (since } I' = (i')^*) \\ &= \langle u, I'u\rangle'_0 \text{ (since } i' \text{ is the inclusion in } G') \\ &= \langle i'u, u\rangle'_0 \text{ (since } i' = (I')^*) \\ &= \langle u, u\rangle'_0 \text{ (since } i' \text{ is the inclusion in } G') \\ &= \langle u, u\rangle_0 \text{ (since } G \leq G') \\ &= \langle iu, u\rangle_0 \text{ (since } i \text{ is the inclusion in } G) \\ &= \langle u, Iu\rangle_0 \text{ (since } i = I^*) \\ &= \langle u, iIu\rangle_0 \text{ (since } i \text{ is the inclusion in } G) \\ &= \langle Iu, Iu\rangle_+ \text{ (since } I = i^*) \\ &= \langle Iu, Iu\rangle'_+ \text{ (since } G \leq G')\end{aligned}$$

Therefore, $\langle I'u - Iu, I'u - Iu\rangle'_+ = \langle I'u, I'u\rangle'_+ - \langle I'u, Iu\rangle'_+ - \langle Iu, I'u'_+\rangle + \langle Iu, Iu\rangle'_+ = 0$, so $I'u = Iu$, i.e.: $I' \restriction H_0 = I$. $\square_{\text{Prop. 5.4.5}}$

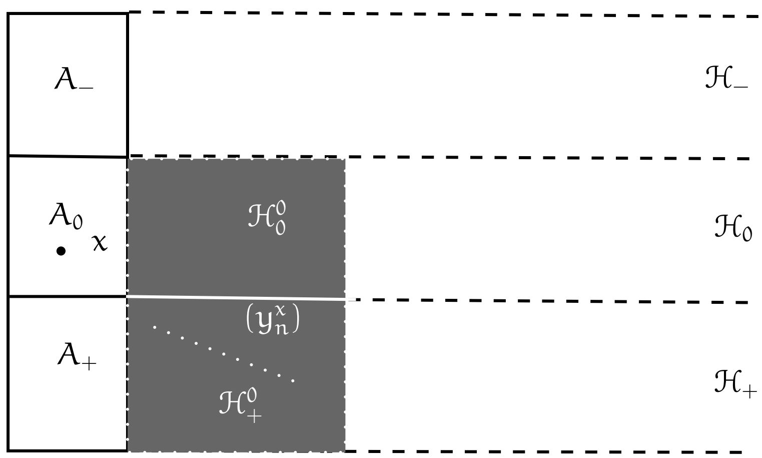
Because of fact 5.4.4 and proposition 5.4.5, in a Gelfand triplet we can drop I as an extra operator.

Proposition 5.4.6. *Let* $\{G_i := (\mathcal{H}^i_+, \mathcal{H}^i_0, \mathcal{H}^i_-) : i < \theta\}$ *be an* $\leq$*-increasing and continuous chain of Gelfand triplets. Then,* $G := (\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ *defined by* $\mathcal{H}_\square := \overline{\bigcup_{i<\theta} \mathcal{H}^i_\square}$ *(*$\square \in \{+, 0, -\}$*) is a Gelfand triplet.*

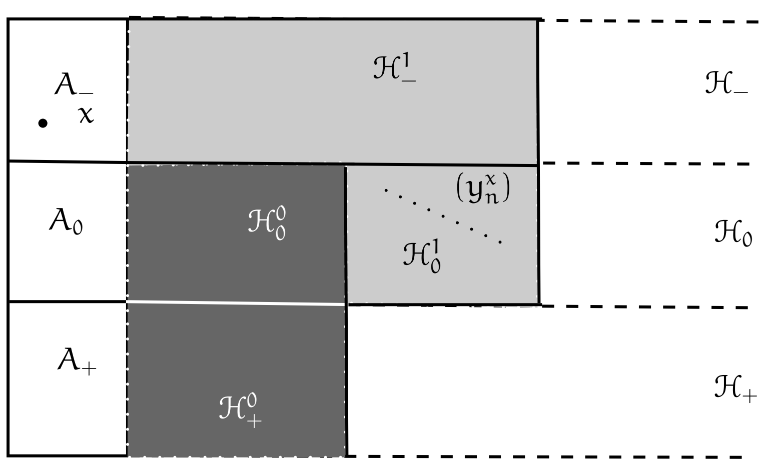
Proof. First, we will prove that the completion of $\mathcal{H}_+$ with the norm $\|\cdot\|_0$ is $\mathcal{H}_0$ and that the completion of $\mathcal{H}_0$ with the norm $\|\cdot\|_-$ is $\mathcal{H}_-$.
Let $x \in \mathcal{H}_0$, so there exists a sequence (x_n) in $\bigcup_{i<\theta} \mathcal{H}^i_0$ such that $(x_n) \to x$ in the norm $\|\cdot\|_0$. Let $i_n < \theta$ be such that $x_n \in \mathcal{H}^{i_n}_0$. Since $\mathcal{H}^{i_n}_+$ is $\|\cdot\|^{i_n}_0$-dense in $\mathcal{H}^{i_n}_0$, there exists a sequence (y^n_k) in $\mathcal{H}^{i_n}_+$ such that $(y^n_k) \to x_n$ in the norm $\|\cdot\|^{i_n}_0$. Let $\varepsilon > 0$ and $B_\varepsilon(x) := \{y \in \mathcal{H}_0 : d_0(y, x) < \varepsilon\}$. Since $(x_n) \to_{\|\cdot\|_0} x$, there exists $N < \omega$ such that for every $n \geq N$ $d_0(x_n, x) < \frac{\varepsilon}{2}$. Since $(y^n_k) \to x_n$ in the norm $\|\cdot\|^{i_n}_0$, then there exists $M < \omega$ such that for every $m \geq M$ $d_0(y^n_m, x_n) = d^{i_n}_0(y^n_m, x_N) < \frac{\varepsilon}{2}$. So, for $k := \max\{M, N\}$, by the triangle inequality we may say $d_0(y^n_k, x) \leq d_0(y^n_k, x_k) + d_0(x_k, x) < \varepsilon$, therefore $\mathcal{H}_+ \cap B_\varepsilon(x) \neq \emptyset$. Then, $\mathcal{H}_+$ is $\|\cdot\|_0$-dense in $\mathcal{H}_0$.
Doing a similar argument, we may say that $\mathcal{H}_0$ is $\|\cdot\|_-$-dense in $\mathcal{H}_-$. $\square_{\text{Prop. 5.4.6}}$

Proposition 5.4.7. *Let* $G := (\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ *be a Gelfand triplet and* $A := (A_+, A_0, A_-) \subset G$. *Then there exists a Gelfand triplet* $G' := (\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-) \leq G$ *such that* $\mathrm{dc}(G') \leq \mathrm{dc}(A) + \aleph_0$ *and* $A \subset G'$.

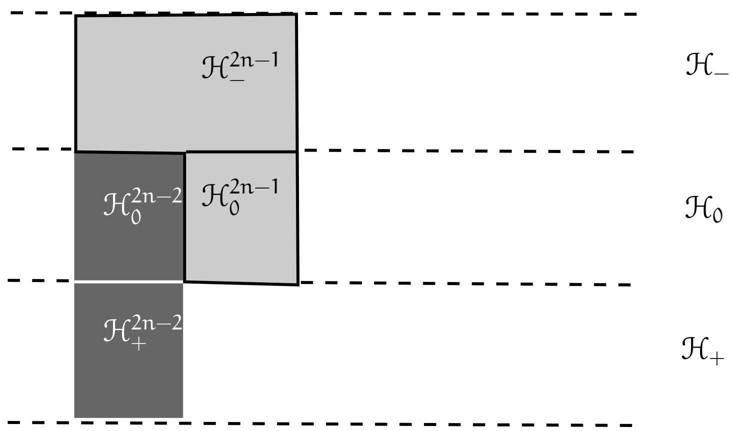
Proof. Let $x \in A_0 \setminus A_+$. Since $x \in H_0$ and G is a Gelfand triplet, there exists a sequence (y^x_n) in $\mathcal{H}_+$ such that $(y^x_n) \to x$ in the norm $\|\cdot\|_0$. If $x \in A_+$, let (y^x_n) be the constant sequence $(x)_{n<\omega}$. Let $\mathcal{H}^0_+ := \overline{\mathrm{span}(y^x_n : x \in A_0, n < \omega)}^+$. Notice that $A_+ \subset \mathcal{H}^0_+$ and $\mathcal{H}^0_+$ is a Hilbert space. Let $\mathcal{H}^0_0 := \overline{\mathcal{H}^0_+}^0$. Notice that $A_0 \subset \mathcal{H}^0_0$ (since if $x \in A_0$ then $x = \lim_{n<\omega} y^x_n \in \mathcal{H}^0_0$, by construction).



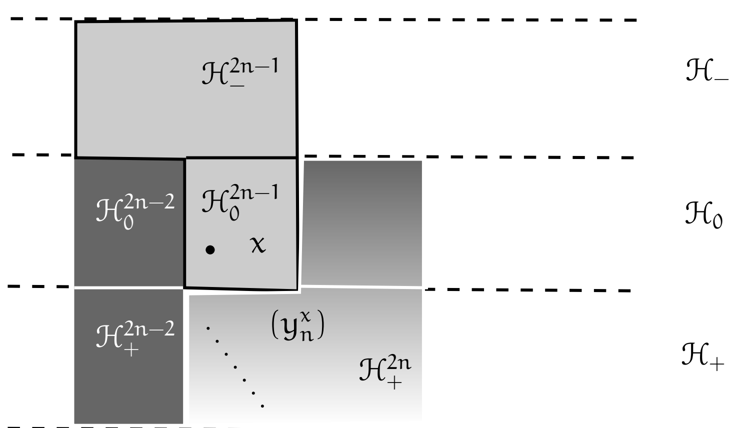
Let $x \in A_-$. Since $x \in \mathcal{H}_-$ and G is a Gelfand triplet, there exists a sequence (y^x_n) in $\mathcal{H}_0$ such that $(y^x_n) \to x$ in the norm $\|\cdot\|_-$. Let $\mathcal{H}^1_0 := \overline{\mathrm{span}(\{y^x_n : x \in A_-, n < \omega\} \cup \mathcal{H}^0_0)}^0$ and $\mathcal{H}^1_- := \left(\overline{\mathcal{H}^1_0}\right)^-$. Notice that $\mathcal{H}^0_0 \subset \mathcal{H}^1_0$ and $A_- \subset \mathcal{H}^1_-$ (if $x \in A_-$, notice that $(y^x_n) \to x \in \mathcal{H}^1_- := \left(\overline{\mathcal{H}^1_0}\right)^-$ since $y^x_n \in H^1_0$ for every $n < \omega$).



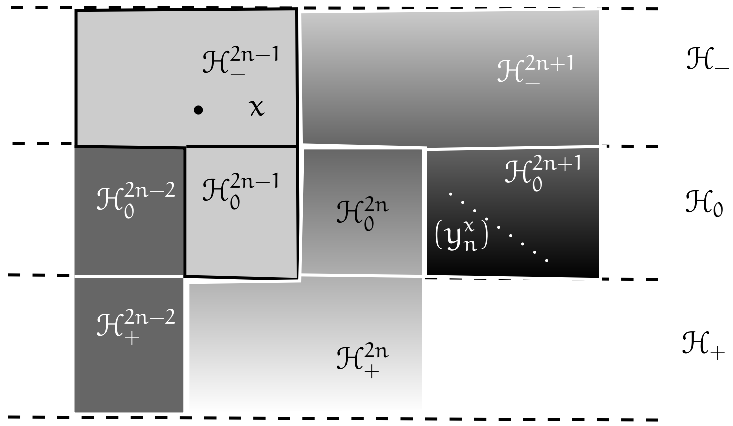
We will build three $\subset$-increasing sequences $\{\mathcal{H}_-^{2n+1} : n < \omega\}$, $\{\mathcal{H}_0^n : n < \omega\}$ and $\{\mathcal{H}_+^{2n} : n < \omega\}$ of Hilbert sub-spaces of $\mathcal{H}_-$, $\mathcal{H}_0$ and $\mathcal{H}_+$ respectively, such that $\mathcal{H}_+^{2n} \subset_{\text{dense}}^{\|\cdot\|_0} \mathcal{H}_0^{2n} \subset \mathcal{H}_0^{2n+1} \subset_{\text{dense}}^{\|\cdot\|_-} \mathcal{H}_-^{2n+1}$.
Let $1 \leq n < \omega$ and suppose we have defined $\mathcal{H}_+^m$, $\mathcal{H}_0^k$ and $\mathcal{H}_-^l$ for every even $m \leq 2n-1$, every $k \leq 2n-1$ and every odd $l \leq 2n-1$. We will define $\mathcal{H}_+^{2n}$, $\mathcal{H}_0^{2n}$, $\mathcal{H}_0^{2n+1}$ and $\mathcal{H}_-^{2n+1}$ satisfying the conditions given above.



Let $x \in \mathcal{H}_0^{2n-1} \setminus \mathcal{H}_+^{2n-2}$. Since $x \in H_0$ and G is a Gelfand triplet, there exists a sequence (y_n^x) in $\mathcal{H}_+$ such that $(y_n^x) \to x$ in the norm $\|\cdot\|_0$. If $x \in \mathcal{H}_+^{2n-2}$, let (y_n^x) be the constant sequence $(x)_{n<\omega}$. Let $\mathcal{H}_+^{2n} := \overline{\mathrm{span}(y_n^x : x \in \mathcal{H}_0^{2n-1}, n < \omega)}^{+}$. Notice that $\mathcal{H}_+^{2n-2} \subset \mathcal{H}_+^{2n}$ and $\mathcal{H}_+^{2n}$ is a Hilbert space. Let $\mathcal{H}_0^{2n} := \overline{\mathcal{H}_+^{2n}}^{0}$. Notice that $\mathcal{H}_0^{2n-1} \subset \mathcal{H}_0^{2n}$ (if $x \in \mathcal{H}_0^{2n-1}$ then $(y_n^x) \to x \in \mathcal{H}_0^{2n} = \overline{\mathcal{H}_+^{2n}}^{0}$, since $y_n^x \in \mathcal{H}_+^{2n}$).



Let $x \in \mathcal{H}_-^{2n-1}$. Since $x \in \mathcal{H}_-$ and G is a Gelfand triplet, there exists a sequence (y_n^x) in $\mathcal{H}_0$ such that $(y_n^x) \to x$ in the norm $\|\cdot\|_-$. Let $\mathcal{H}_0^{2n+1} := \overline{\mathrm{span}(\{y_n^x : x \in \mathcal{H}_-^{2n-1}, n < \omega\} \cup \mathcal{H}_0^{2n})}^{0}$ and $\mathcal{H}_-^{2n+1} := \left(\overline{\mathcal{H}_0^{2n+1}}\right)^{-}$. Notice that $\mathcal{H}_0^{2n} \subset \mathcal{H}_0^{2n+1}$ (by construction) and $\mathcal{H}_-^{2n-1} \subset \mathcal{H}_-^{2n+1}$ (if $x \in \mathcal{H}_-^{2n-1}$, notice that $(y_n^x) \to x \in \mathcal{H}_-^{2n+1} := \left(\overline{\mathcal{H}_0^{2n+1}}\right)^{-}$ because $y_n^x \in \mathcal{H}_0^{2n+1}$ for every $n < \omega$).



Let $\mathcal{H}'_+ := \overline{\bigcup_{n<\omega} \mathcal{H}_+^{2n}}^{+}$, $\mathcal{H}'_0 := \overline{\bigcup_{n<\omega} \mathcal{H}_0^{n}}^{0}$ and $\mathcal{H}'_- := \left(\overline{\bigcup_{n<\omega} \mathcal{H}_-^{2n+1}}\right)^{-}$.

Notice that $\mathcal{H}'_+ \subset \mathcal{H}'_0 \subset \mathcal{H}'_-$: Let $x \in \mathcal{H}'_+$, so there exists a sequence (x_n) in $\bigcup_{n<\omega} \mathcal{H}_+^{2n}$ such that $(x_n) \to x$ in the norm $\|\cdot\|_+$. Notice that there exists $k_n < \omega$ such that $x_n \in \mathcal{H}_+^{2k_n}$, therefore $x_n \in \mathcal{H}_0^{2k_n}$, so (x_n) is a sequence in $\mathcal{H}'_0$. Since $(x_n) \to x$ in the norm $\|\cdot\|_+$, then $(x_n) \to x$ in the norm $\|\cdot\|_0$ (because $\|u\|_0 \leq \|u\|_+$ for all $u \in H_+$), then $x \in \mathcal{H}'_0$. Let $x \in \mathcal{H}'_0$, then there exists a sequence (x_n) in $\bigcup_{n<\omega} \mathcal{H}_0^{n}$ such that $(x_n) \to x$ in the norm $\|\cdot\|_0$. Let $k_n < \omega$ such that $x_n \in \mathcal{H}_0^{k_n}$. Notice that if k_n is odd then $x_n \in \mathcal{H}_-^{k_n}$ and if k_n is even then $x_n \in \mathcal{H}_-^{k_n+1}$. Anyway, $x_n \in \mathcal{H}'_-$. Since $(x_n) \to x$ in the norm $\|\cdot\|_0$, then $(x_n) \to x$ in the norm $\|\cdot\|_-$ (because $\|u\|_- \leq \|u\|_0$ for all $u \in H_0$). Therefore, $x \in \mathcal{H}'_-$.
Moreover, $\mathcal{H}'_+ \subset^{\|\cdot\|_0}_{\text{dense}} \mathcal{H}'_0 \subset^{\|\cdot\|_-}_{\text{dense}} \mathcal{H}'_-$: Let $x \in \mathcal{H}'_-$ and $\varepsilon > 0$, therefore there exists a sequence (x_n) in $\bigcup_{n<\omega} \mathcal{H}_-^{2n+1}$ such that $(x_n) \to x$ in the norm $\|\cdot\|_-$. So, there exists $N < \omega$ such that for every $n \geq N$ $d_-(x, x_n) < \varepsilon/2$. Given $n \geq N$, there exists $k_n < \omega$ such that $x_n \in \mathcal{H}_-^{2k_n+1}$. By construction, there exists a sequence $(y_m^{x_n})$ in $\mathcal{H}_0^{2k_n+3}$ such that $(y_m^{x_n}) \to x_n$ in the norm $\|\cdot\|_-$. Therefore, there exists $M < \omega$ such that for every $m \geq M$ $d_-(x_n, y_m^{x_n}) < \varepsilon/2$. So, $d_-(x, y_m^{x_n}) \leq d_-(x, x_n) + d_-(x_n, y_m^{x_n}) < \varepsilon$. Therefore, the ball $B_\varepsilon(x) := \{y \in \mathcal{H}'_- : d_-(x, y) < \varepsilon\}$ has an element in $\mathcal{H}'_0$; and so $\mathcal{H}'_0$ is $\|\cdot\|_-$-dense in $\mathcal{H}'_-$.
Let $x \in \mathcal{H}'_0$ and $\varepsilon > 0$, therefore there exists a sequence (x_n) in $\bigcup_{n<\omega} \mathcal{H}_0^{n}$ such that $(x_n) \to x$ in the norm $\|\cdot\|_0$. So, there exists $N < \omega$ such that for every $n \geq N$ $d_0(x, x_n) < \varepsilon/2$. Given $n \geq N$, there exists $k_n < \omega$ such that $x_n \in \mathcal{H}_0^{k_n}$. By construction, if k_n is even there exists a sequence $(y_m^{x_n})$ in $\mathcal{H}_+^{k_n+2}$ such that $(y_m^{x_n}) \to x_n$ in the norm $\|\cdot\|_0$, if k_n is odd there exists a sequence $(y_m^{x_n})$ in $\mathcal{H}_+^{k_n+1}$ such that $(y_m^{x_n}) \to x_n$ in the norm $\|\cdot\|_0$. Anyway, such that sequence $(y_m^{x_n})$ is in $\mathcal{H}'_+$. Therefore, there exists $M < \omega$ such that for every $m \geq M$ $d_0(x_n, y_m^{x_n}) < \varepsilon/2$. So, $d_0(x, y_m^{x_n}) \leq d_0(x, x_n) + d_0(x_n, y_m^{x_n}) < \varepsilon$. Therefore, the ball $B_\varepsilon(x) := \{y \in \mathcal{H}'_0 : d_0(x, y) < \varepsilon\}$ has an element in $\mathcal{H}'_+$; and so $\mathcal{H}'_+$ is $\|\cdot\|_0$-dense in $\mathcal{H}'_0$.
Therefore, G' is a Gelfand triplet such that $A \subset G' \leq G$. $\square_{\text{Prop. 5.4.7.}}$

Proposition 5.4.8 (Amalgamation Property). *If $\mathcal{H}^0 \leq \mathcal{H}^1, \mathcal{H}^2$ are Gelfand triplets, then there exist a Gelfand triplet $\mathcal{H}$ and $\leq$-embeddings $f_i : \mathcal{H}^i \to \mathcal{H}$ which fix $\mathcal{H}^0$ pointwise.*

Proof. Let $(\mathcal{H}^0_\square)^{\perp_i}$ be the perpendicular space to $\mathcal{H}^0_\square$ inside $\mathcal{H}^i_\square$ ($\square \in \{+,0,-\}$ and $i \in \{1,2\}$). So $H^0_\square \oplus (\mathcal{H}^0_\square)^{\perp_i} \approx H^i_\square$. Define $\mathcal{H}_\square := \mathcal{H}^0_\square \oplus (\mathcal{H}^0_\square)^{\perp_1} \oplus (\mathcal{H}^0_\square)^{\perp_2}$. We will see that $\mathcal{H} := (\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ is a Gelfand triplet:

Let $x_- := (x^0_-, x^1_-, x^2_-) \in \mathcal{H}_-$ and $(y^i_n)_n \in \mathcal{H}^i_0$ be a sequence such that $(y^i_n)_n \to_{\|\cdot\|_-} x^i_-$ ($i \in \{0,1,2\}$). Let $\varepsilon > 0$. So, there exists $N_i < \omega$ such that for every $n \geq N_i$ $\|x^i_- - y^i_n\|_- < \frac{\varepsilon\sqrt{3}}{3}$, so $\langle x^i_- - y^i_n, x^i_- - y^i_n \rangle_- < \varepsilon^2/3$. Let $y_n := (y^0_n, y^1_n, y^2_n)$. Therefore, if $n \geq \max\{N_0, N_1, N_2\}$, then $\langle x_- - y_n, x_- - y_n \rangle_- = \langle x^0_- - y^0_n, x^0_- - y^0_n \rangle_- + \langle x^1_- - y^1_n, x^1_- - y^1_n \rangle_- + \langle x^2_- - y^2_n, x^2_- - y^2_n \rangle_- < 3(\varepsilon^2/3) = \varepsilon^2$, so $\|x_- - y_n\|_- < \varepsilon$. Hence $x_- \in (\overline{\mathcal{H}_0})^-$, and so $\mathcal{H}_- = (\overline{\mathcal{H}_0})^-$.

Let $x_0 := (x^0_0, x^1_0, x^2_0) \in \mathcal{H}_0$ and $(y^i_n)_n \in \mathcal{H}^i_+$ be a sequence such that $(y^i_n)_n \to_{\|\cdot\|_0} x^i_0$ ($i \in \{0,1,2\}$). Let $\varepsilon > 0$. So, there exists $N_i < \omega$ such that for every $n \geq N_i$ $\|x^i_0 - y^i_n\|_0 < \frac{\varepsilon\sqrt{3}}{3}$, so $\langle x^i_0 - y^i_n, x^i_0 - y^i_n \rangle_0 < \varepsilon^2/3$. Let $y_n := (y^0_n, y^1_n, y^2_n)$. Therefore, if $n \geq \max\{N_0, N_1, N_2\}$, then $\langle x_0 - y_n, x_0 - y_n \rangle_0 = \langle x^0_0 - y^0_n, x^0_0 - y^0_n \rangle_0 + \langle x^1_0 - y^1_n, x^1_0 - y^1_n \rangle_0 + \langle x^2_0 - y^2_n, x^2_0 - y^2_n \rangle_0 < 3(\varepsilon^2/3) = \varepsilon^2$, so $\|x_0 - y_n\|_0 < \varepsilon$. Hence $x_0 \in (\overline{\mathcal{H}_+})^0$, and so $\mathcal{H}_0 = (\overline{\mathcal{H}_+})^0$. $\square_{\text{Prop. 5.4.8}}$

Proposition 5.4.9 (Joint Embedding Property). *Let $\mathcal{H}^1, \mathcal{H}^2$ be Gelfand triplets. Then there exists a Gelfand triplet $\mathcal{H}$ and $\leq$-embeddings $f_i : \mathcal{H}^i \to \mathcal{H}$.*

Proof. Let $\mathcal{H}_\square := \mathcal{H}^1_\square \oplus \mathcal{H}^2_\square$ ($\square \in \{+,0,-\}$).

Notice that $\mathcal{H} := (\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ is a Gelfand triplet: Let $x := (x^1, x^2) \in \mathcal{H}_-$. We know that here exists a sequence $(x^i_n)_n \in \mathcal{H}^i_0$ such that for every $\varepsilon > 0$ there exists $N_i < \omega$ such that for every $n \geq N_i$ $d(x^i_n, x^i) = \|x^i_n - x^i\|_- = \sqrt{\langle x^i_n - x^i, x^i_n - x^i \rangle_-} < \frac{\varepsilon\sqrt{2}}{2}$. So, if $n \geq \max\{N_1, N_2\}$ then $\langle x_n - x, x_n - x \rangle_- = \langle x^1_n - x^1, x^1_n - x^1 \rangle_- + \langle x^2_n - x^2, x^2_n - x^2 \rangle_- < 2(\varepsilon^2/2) = \varepsilon^2$, so $((x^1_n, x^2_n)) \to_{\|\cdot\|_-} (x^1, x^2)$. Therefore, $(\overline{\mathcal{H}_0})^- = \mathcal{H}_-$.

Let $x := (x^1, x^2) \in \mathcal{H}_0$. We know that here exists a sequence $(x^i_n)_n \in \mathcal{H}^i_+$ such that for every $\varepsilon > 0$ there exists $N_i < \omega$ such that for every $n \geq N_i$ $d(x^i_n, x^i) = \|x^i_n - x^i\|_0 = \sqrt{\langle x^i_n - x^i, x^i_n - x^i \rangle_0} < \frac{\varepsilon\sqrt{2}}{2}$. So, if $n \geq \max\{N_1, N_2\}$ then $\langle x_n - x, x_n - x \rangle_0 = \langle x^1_n - x^1, x^1_n - x^1 \rangle_0 + \langle x^2_n - x^2, x^2_n - x^2 \rangle_0 < 2(\varepsilon^2/2) = \varepsilon^2$, so $((x^1_n, x^2_n)) \to_{\|\cdot\|_0} (x^1, x^2)$. Therefore, $(\overline{\mathcal{H}_+})^0 = \mathcal{H}_0$. $\square$

Bibliography

[AB09] C. Argoty and A. Berenstein. Model theory of Hilbert spaces with unitary operators. *Math. Logic Quaterly*, 55(1):1–12, 2009.

[Arg1X] C. Argoty. Model theory of a Hilbert space expanded with an unbounded closed selfadjoint operator. Available in http://arxiv.org/pdf/1102.2454v2, 201X.

[Bal09] J. Baldwin. Categoricity. volume 50 of *University Lecture Series*. American Mathematical Society, 2009.

[Bal0x] J. Baldwin. Splitting independence in superstable AEC. 200x.

[BBH09] I. BenYaacov, A. Berenstein, and W. Henson. Model theoretic independence in the Banach lattices $L^p(\mu)$. Available in http://arxiv.org/abs/0907.5273, 2009.

[BH04] A. Berenstein and W. Henson. Model theory of probability spaces with an automorphism. Avalaible at http://xxx.lanl.gov/pdf/math/0405360v1, 2004.

[BKV06] J. Baldwin, D. Kueker, and M. VanDieren. Upward stability transfer theorem for tame abstract elementary classes. *Notre Dame J. Formal Logic*, 47(2):291–298, 2006.

[BS91] J. Baldwin and S. Shelah. The primal framework II: Smoothness. *APAL*, 55:1–34, 1991. Shelah's publication n. 360.

[Bue85] S. Buechler. The geometry of weakly minimal types. *Journal of Symbolic Logic*, 50(4):1044–1053, 1985.

[Bue96] S. Buechler. *Essential Stability Theory*. Springer - Verlag, 1996.

[BY03] I. Ben-Yaacov. Positive model theory and compact abstract theories. *J. Math. Logic*, 3(1):85–118, 2003.

[BY05] I. Ben-Yaacov. Uncountable dense categoricity in cats. *J. Symbolic Logic*, 70(3):829–860, 2005.

[BYBHU08] I. Ben-Yaacov, A. Berenstein, W. Henson, and A. Usvyatsov. Model theory for metric structures. In *Model theory with applications to algebra*, volume 2 of *Lecture Note Series no. 350*. London Mathematical Society, 2008.

[CK66] C. C. Chang and H. J. Keisler. *Continuous model theory*. Number 58 in Annals of Mathematics Studies. Princeton Univ. Press, 1966.

[Gro02] R. Grossberg. Classification theory for abstract elementary classes. In *Logic and algebra*, volume 302 of *Contemporary Mathematics*. American Mathematical Society, 2002.

[GV06a] R. Grossberg and M. VanDieren. Categoricity from one successor cardinal in tame abstract elementary classes. *J. Math. Logic*, 6(2):181–201, 2006.

[GV06b] R. Grossberg and M. VanDieren. Galois-stability for tame abstract elementary classes. *J. Math. Logic*, 6(1):25–49, 2006.

[GVV08] R. Grossberg, M VanDieren, and A. Villaveces. Uniqueness of limit models in classes with amalgamation. http://www.math.cmu.edu/~rami/gvv_11_5.08.pdf, 2008.

[HH09] Å. Hirvonen and T. Hyttinen. Categoricity in homogeneous complete metric spaces. *Arch. Math. Logic*, 48:269–322, 2009.

[HH11] Å. Hirvonen and T. Hyttinen. Metric abstract elementary classes with perturbations. Reports in Mathematics, University of Helsinki. Report number 517., 2011.

[HI02] C. W. Henson and J. Iovino. Ultraproducts in analysis. In *Analysis and Logic*, volume 262 of *London Math. Soc. Lecture Note Series*. Cambridge University Press, 2002.

[Hir06] Å. Hirvonen. Categoricity in metric abstract elementary classes. Ph.L. thesis, University of Helsinki, 2006.

[HK06] T. Hyttinen and M. Kesäla. Categoricity transfer in simple finitary abstract elementary classes. Part of M. Kesäla's Ph.D. thesis at University of Helsinki., 2006.

[Hru86] E. Hrushovski. *Contributions to stable model theory*. PhD thesis, University of California at Berkeley, 1986.

[Jón56] B. Jónsson. Universal relational systems. *Mathematica Scandinavica*, 4:193–208, 1956.

[Jón60] B. Jónsson. Homogeneous universal relational systems. *Mathematica Scandinavica*, 8:137–142, 1960.

[KC66] H. J. Keisler and C. C. Chang. *Continuous model theory*. Princeton University Press, 1966.

[Lim93] E. L. Lima. *Espaços métricos*. IMPA, 1993.

[Mad] R. de la Madrid. The role of rigged Hilbert space in quantum mechanics. Available at http://arxiv.org/abs/quant-ph/0502053.

[Mak85] J. A Makowsky. Abstract embedding relations. In J. Barwise and S. Feferman, editors, *Model - theoretic Model-theoretic logics*. Springer - Verlag, 1985.

[Mor65] M. Morley. Omiting classes of elements. In Henkin Addison and Tarski, editors, *The theory of models*, pages 265–273. North Holland, 1965.

[OU09a] A. Onshuus and A. Usvyatsov. Orthogonality and domination in unstable theories. Available at http://www.math.ucla.edu/~alexus/papers/th-weight.pdf, 2009.

[OU09b] A. Onshuus and A. Usvyatsov. Stable domination and weight. Available at http://www.math.ucla.edu/~alexus/papers/domination-11-06.pdf, 2009.

[Pil96] A. Pillay. *Geometric Stability Theory*. Clarendon Press, 1996.

[Poi06] Pedro Poitevin. *Model theory of Nakano spaces*. PhD thesis, University of Illinois at Urbana-Champaigne, 2006.

[She] S. Shelah. When first order T has limit models. Shelah's publication no. 868.

[She75] S. Shelah. The lazy model-theoretician's guide to stability. *Logique et Analyse*, 18:241–308, 1975.

[She87a] S. Shelah. Classification of nonelementary classes II, abstract elementary classes. In J. Baldwin, editor, *Proceedings of the USA-Israel Conference on Classification Theory, Chicago, December 1985*, volume 1292 of *Lecture Notes in Mathematics*, pages 419–497. Springer-Verlag, 1987. Shelah's publitacion no. 88.

[She87b] S. Shelah. Universal classes, part I. In J. Baldwin, editor, *Proceedings of the USA-Israel Conference on Classification Theory, Chicago, December 1985*, volume 1292 of *Lecture Notes in Mathematics*, pages 264–419. Springer-Verlag, 1987. Shelah's publitacion no. 300.

[She99] S. Shelah. Categoricity for abstract classes with amagalmation. *APAL*, 98:261–294, 1999.

[She09a] S. Shelah. Categoricity in abstract elementary classes: Going up inductively. In *Classification theory for Abstract Elementary Classes*, volume 18 of *Studies in Logic – Mathematical Logic and Foundations*, pages 224–377. College Publications, 2009. Shelah's publitacion no. 600.

[She09b] S. Shelah. Toward classification theory of good λ-frames and abstract elementary classes. In *Classification theory for Abstract Elementary Classes*, volume 18 of *Studies in Logic – Mathematical Logic and Foundations*, pages 378–644. College Publications, 2009. Shelah's Publication no 705.

[SS78] J. Stern and S. Shelah. The Hanf number of the first order theory of Banach spaces. *Transactions american Math. Soc.*, 244:147–171, 1978. Shelah's publication n. 63.

[SU] S. Shelah and A. Usvyatsov. Model theoretic stability and categoricity for complete metric spaces. Shelah's publication n. 837.

[SV99] S. Shelah and A. Villaveces. Toward categoricity for classes with no maximal models. *APAL*, 97:1–25, 1999. Shelah's publication n. 635.

[TZ09] K. Tent and M. Ziegler. A course in model theory. U. Müster and U. Freiburg, 2009.

[Van] M. VanDieren. Limit models in abstract elementary classes. Talk given in Bogota Model Theory Meeting, 2007.

[Van06] M. VanDieren. Categoricity in abstract elementary classes with no maximal models. *APAL*, 141(1-2):108–147, 2006.

[VZ10a] A. Villaveces and P. Zambrano. Around stability in metric abstract elementary classes: r-towers and limit models. *Mittag-Leffler instiute preprint series*, 2010.

[VZ10b] A. Villaveces and P. Zambrano. Notions of independence in metric abstract elementary classes. *Mittag-Leffler institute preprint series*, 2010.

[VZ1x] A. Villaveces and P. Zambrano. Around stability in metric abstract elementary classes: existence and uniqueness of limit models. Preprint, 201x.

[Wag00] F. Wagner. *Simple theories*. Mathematics and its applications. Kluwer Academic Publishers, 2000.

[Wil70] A. Wilansky. *Topology for analysis*. Ginn and company, 1970.

[Wlo73] J. Wloka. Gelfand triplets and spectral theory. volume 331 of *Lecture Notes in Mathematics*, pages 163–182. Springer-Verlag, 1973.

[Zam1x] Pedro Zambrano. A stability transfer theorem in d-tame abstract elementary classes. Submitted., 201x.

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