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Estos años que ahora se cierran entre lo prosáico y lo poético, con el nombre genérico de lo matemático, *me recuerdan a una de esas muchachas endebles y enfermizas a las que a veces se mira con lástima, a veces con una especie de afecto compasivo, y a veces, sencillamente, no se fija uno en ellas, pero que de pronto, en un abrir y cerrar de ojos, sin que se sepa cómo, se convierten en beldades singulares y prodigiosas. Y uno, asombrado, cautivado, se pregunta sin más: ¿qué impulso ha hecho brillar con tal fuego esos ojos tristes y pensativos?, ¿qué ha hecho volver la sangre a esas mejillas pálidas y sumidas?, ¿qué ha regado de pasión los rasgos de ese tierno rostro?, ¿de qué palpita ese pecho?, ¿qué ha traído de súbito vida, vigor y belleza al rostro de la pobre muchacha?, ¿qué la ha hecho iluminarse con tal sonrisa, animarse con esa risa cegadora y chispeante? Mira uno en torno suyo buscando a alguien, sospechando algo. Pero pasa ese momento y quizás al día siguiente encuentra uno la misma mirada vaga y pensativa de antes, el mismo rostro pálido, la misma humildad y timidez en los movimientos; y más aún: remordimiento, rastros de cierta torva melancolía y aun irritación ante el momentáneo enardecimiento. Y le apena a uno que esa instantánea belleza se haya marchitado de manera tan rápida e irrevocable, que haya brillado tan engañosa e ineficazmente ante uno; le apena el que ni siquiera hubiese tiempo bastante para enamorarse de ella..*

Fiódor Dostoievski en mis *Noches blancas*.

Abstract

The aim of this work is to study certain type of operators that have adquired a renewed relevance in the last decade. This wide class of operators, genericaly called *nonlocal* operators, appears naturally in many applications in physics, probability, biology and economics.

Throughout this work our attention will be focused in a very important nonlocal operator known as the *fractional Laplacian*. This operator plays a similar role in the theory of nonlocal operators as the one played by the Laplacian in the classical theory of elliptic partial differential equations, i.e. the fractional Laplacian constitutes the most simple and, at the same time, the "model" nonlocal operator and therefore it is considered as the benchmark in the study of these kind of operators. Thus, by analogy, most of the relevant questions associated to the Laplacian (e.g. comparison principles, linear and nonlinear boundary value problems, and regularity results) have an equivalent for the fractional Laplacian. Also, in the same spirit of the multiplicity results for the classical semilinear partial differential equations we prove a multiplicity result for semilinear problems involving the fractional Laplacian using standard variational methods.

Keywords: Nonlocal operators, fractional Laplacian, semilinear PDE, variational methods.

Introduction

The aim of this work is to study certain type of operators that have adquired a renewed relevance in the last decade. This wide class of operators, genericaly called *nonlocal* operators, appears naturally in many applications in physics, probability, biology and economics. Also they are relevant by themselves in mathematics because, as we will discusse in this work (see section 2.1 below), every linear operator that satisfies the maximum principle (i.e. "distinguish maximizers" of smooth functions) can be decomposed into a local operator and a nonlocal operator.

Roughly speaking, we can understand a nonlocal operator as a functional whose output or value depends on the whole domain of the input or argument. This characteristic is usually translated in the applications as phenomena that involve, for instance, the interaction of bacteria, economic agents, layered materials and so forth, whose individual or local reaction to an external force depends on the reaction of all the other components of the system.

Throughout this work our attention will be focused in a very important nonlocal operator known as the *fractional Laplacian*. This operator plays a similar role in the theory of nonlocal operators as the one played by the Laplacian in the classical theory of elliptic partial differential equations, i.e. the fractional Laplacian constitutes the most simple and, at the same time, the "model" nonlocal operator and therefore it is considered as the benchmark in the study of these kind of operators. Thus, by analogy, most of the relevant questions associated to the Laplacian (e.g. comparison principles, linear and nonlinear boundary value problems, and regularity results) have an equivalent for the fractional Laplacian.

As will be better explained below, there are several ways to introduce the fractional Laplacian. For instance, using properties of Fourier transform ($\mathscr{F}$), if u is an adequate function (e.g. u lies in the Schwartz space $\mathscr{S}(\mathbb{R}^N)$),

$$\mathscr{F}(\Delta u)(\xi) = -4\pi^2|\xi|^2\mathscr{F}(u),$$

and so

$$-\Delta u = 4\pi^2\mathscr{F}^{-1}(|\xi|^2\mathscr{F}(u)).$$

Motivated by this observation, one could define for for $s \in (0,1)$ and for $u \in \mathscr{S}(\mathbb{R}^N)$

$$(-\Delta u)^s := (2\pi)^{2s}\mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}(u)).$$

This definition can be extended to larger spaces (see chapter 4.).

In this work we are principally interested in studying the basic properties of the fractional Laplacian to adquire the necessary tools that will allow us to understand and approach certain types of nonlinear boundary value problems. Keeping in mind this purpose, and for the sake of clarity in the exposition, we chose the following outline. Chapter 1. provides the required mathematical background like Sobolev spaces, Fourier transform and some basic properties of these two mathematical objects, besides, this chapter introduces several notations that will be used in the rest of the thesis. In chapter 2. we present three different approaches to motivate the introduction of the nonlocal operators and, more precisely, the fractional Laplacian. Chapter 3. is devoted to Fractional Sobolev Spaces, which generalize the classical Sobolev Spaces, seeking to introduce the concept of fractional derivatives. Chapter 4. contains a rigorous treatment of our main operator, some equivalent definitions and its basic properties. In chapter 5. we address the nonlocal analogous to the Poisson problem with Dirichlet boundary condition and we study the spectrum of the fractional Laplacian. Chapter 6. contains the main contribution of this work in which motivated by the results of [4] and [5] in the local case, we prove two multiplicity results for the nonlinear and *nonlocal boundary* value problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega^c, \end{cases}$$

with an asymptotically linear reaction term f (see chapter 6. below for precise statements). We point out that these results are a part of a work in progress that intends to extend and generalize the results of the aforementioned references (and from some other, by the way) to the nonlocal setting.

Finally, at the end of chapter 6. we have included an index of notation.

Contents

1. Preliminaries

1.1 Function spaces

Throughout this thesis N will be a fixed natural number, besides, unless we specify otherwise, any generic function u will be real-valued and defined on a subset A of $\mathbb{R}^N$. Therefore, given a measurable subset $\Omega \subset \mathbb{R}^N$ and $p \in [1,\infty]$, the space of measurable functions defined on Ω, denoted by $\mathscr{M}(\Omega)$, and its subspaces of integrable functions, $L^p(\Omega)$, are understood to contain exclusively real-valued functions.

Definition 1 (Multi-index notation)**.** *We define the set of multi-indices in $\mathbb{R}^N$ as $\mathbb{N}_0^N$ with* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
For each multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ we introduce the following notation:

- $|\alpha| := \sum_{i=1}^N \alpha_i$. *This quantity is called "the length of the multi-index α".*
- $x^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ *for all $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$.*
- *Given a function u, $|\alpha|$-times differentiable at some point $x \in \mathbb{R}^N$ we denote*

  the α-derivative of u as follows:

$$\partial^\alpha u(x) := \frac{\partial^{\alpha_N} \dots \partial^{\alpha_1}}{\partial x_N^{\alpha_N} \dots \partial x_1^{\alpha_1}} u(x),$$

  with $\partial^0 u(x) := u(x)$.

Remark 2. *For derivatives of low order (first and second order) we will keep the notation:*

$$\partial_i u(x) := \frac{\partial_i u}{\partial x_i}(x), \qquad \partial_{ji} u(x) := \frac{\partial^2 u}{\partial x_j \partial x_i}(x), \qquad \forall i, j \in \{1, \dots, N\}.$$

Definition 3. *Let Ω be an open subset of $\mathbb{R}^N$. We define the space of k times continuous differentiable real-valued functions up to the boundary as*

$$C^k(\overline{\Omega}) := \{u \in C^k(\Omega) |\ \partial^\alpha u \text{ admits a continuous extension to } \overline{\Omega},\ \forall\, \alpha \in \mathbb{N}_0^N, |\alpha| \le k\},$$

and the space of bounded k times continuous differentiable real-valued functions up to the boundary as

$$C_B^k(\overline{\Omega}) := \{u \in C^k(\Omega) |\ \partial^\alpha u \text{ is bounded and admits a continuous extension to } \overline{\Omega},\ \forall\, \alpha \in \mathbb{N}_0^N, |\alpha| \le k\},$$

endowed with the norm $\|u\|_{C_B^k(\overline{\Omega})} := \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^\infty(\overline{\Omega})}$.
In the same spirit we define the spaces

$$\begin{aligned} C_0^k(\Omega) &:= \{u \in C^k(\overline{\Omega}) |\ supp(u) \subset\subset \Omega\}, \\ C_B^\infty(\overline{\Omega}) &:= \bigcap_{k \in \mathbb{N}} C_B^k(\overline{\Omega}), \\ C_0^\infty(\Omega) &:= \bigcap_{k \in \mathbb{N}} C_0^k(\Omega). \end{aligned}$$

Definition 4. *Let $\Omega \subset \mathbb{R}^N$ and $\gamma \in (0,1]$. We say that $u : \Omega \to \mathbb{R}$ is a γ-Hölder continuous function if there exists a constant $k > 0$ such that*

$$|u(x) - u(y)| \leq k|x-y|^\gamma, \qquad \forall x, y \in \Omega.$$

Under this consideration, if Ω is open, $k \in \mathbb{N}_0$ and $\gamma \in (0,1]$, we define the vector space

$$C^{k,\gamma}(\Omega) := \{u \in C^k(\Omega) |\ \partial^\alpha u \text{ is } \gamma\text{-Hölder continuous on } \Omega,\ \forall\, \alpha \in \mathbb{N}_0^N, |\alpha| = k\},$$

and the normed vector spaces

$$C_B^{k,\gamma}(\Omega) := \{u \in C^k(\Omega) |\ \partial^\alpha u \text{ is bounded and } \gamma\text{-Hölder continuous on } \Omega,\ \forall \alpha \in \mathbb{N}_0^N |\alpha| = k\},$$
$$C_0^{k,\gamma}(\Omega) := C^{k,\gamma}(\Omega) \cap C_0^k(\Omega),$$

both endowed with the norm:

$$\|u\|_{C_B^{k,\gamma}(\Omega)} := \|u\|_{C_B^k(\Omega)} + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\gamma}.$$

Definition 5 (Weak derivative)**.** *Let Ω be an open subset of $\mathbb{R}^N$. Given $u \in L^1_{loc}(\Omega)$, we say that $u_i \in L^1_{loc}(\Omega)$ is a weak derivative of u in the direction $i \in \{1, \ldots, N\}$ if*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega u_i \phi, \qquad \forall \phi \in C_0^1(\Omega). \tag{1.1}$$

The following proposition summarizes some basic properties of the weak derivatives.

Proposition 6. *Let Ω be an open subset of $\mathbb{R}^N$ and let $1 \leq p \leq \infty$. Given $u \in L^p(\Omega)$ and $i \in \{1, \ldots, N\}$.*

(i) *If u has two weak derivatives u_i and v_i in the direction i then $u_i = v_i$ a.e. in Ω.*

(ii) *If $\partial_i u$ exists in the classical sense and $\partial_i u \in L^1_{loc}(\Omega)$ then $\partial_i u$ is the weak derivative of u in the direction i.*

Proof. See chapters 8 and 9 in [2]. □

Remark 7. *The last result shows us that, in some sense, the notion of weak differentiability generalizes the classical notion of differentiability. Hence, when there is no confusion we will use the same notation of the classical derivatives for the weak derivatives.*

Definition 8 (Sobolev spaces)**.** *Let Ω be an open subset of $\mathbb{R}^N$ and let $1 \leq p \leq \infty$. For $k \in \mathbb{N}$ with $k > 1$ we define the Sobolev spaces inductively as follows:*

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) |\quad \text{the weak derivative } \partial_i u \text{ exists and } \partial_i u \in L^p(\Omega),\ \forall i = 1, \ldots N\}.$$
$$W^{k,p}(\Omega) := \{u \in W^{k-1,p}(\Omega) |\quad \partial_i u \in W^{k-1,p}(\Omega),\ \forall i = 1, \ldots N\},$$

In the following definition we introduce the local model for open sets with regular boundary.

Definition 9. *Given $N \geq 2$ we define sets*

$$Q := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \big| \quad |x'| < 1, |x_N| < 1\},$$

$$Q_+ := \{x = (x', x_N) \in Q \big| \quad x_N > 0\} \quad and \quad Q_0 := \{x = (x', x_N) \in Q \big| \quad x_N = 0\}.$$

Definition 10. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $N \geq 2$, $k \in \mathbb{N}_0$ and $\gamma \in (0,1]$. We say that Ω is an open set of class $C^{k,\gamma}$ if for every $x \in \partial\Omega$ there exist $r > 0$ and a homeomorphism $\Psi : Q \to B_r(x)$ such that*

$$\Psi \in C_B^{k,\gamma}(Q), \quad \Psi^{-1} \in C_B^{k,\gamma}(B_r(x)), \quad \Psi(Q_+) = B_r(x) \cap \Omega, \quad \Psi(Q_0) = B_r(x) \cap \partial\Omega.$$

Now we present a key theorem that allow us to translate global problems into local problems.

Theorem 11. *Let $\mathscr{A}$ be a collection of open sets in $\mathbb{R}^N$ and let A be their union. There exist $M \subset \mathbb{N}$ and a family $\{\phi_i\}_{i\in M}$ of continuous functions $\phi_i : \mathbb{R}^N \to \mathbb{R}$ such that*

(i) *$\phi_i(x) \geq 0$ for all $x \in \mathbb{R}^N$ and for all $i \in M$.*

(ii) *$S_i := \operatorname{supp} \phi_i \subset A$ for all $i \in M$.*

(iii) *Each point $x \in A$ has a neighbourhood that intersects only finitely many sets S_i.*

(iv) *$\sum_{i\in M} \phi_i(x) = 1$ for all $x \in A$.*

(v) *$\phi_i \in C_0^\infty(\mathbb{R}^N)$ for all $i \in M$.*

(vi) *S_i is contained in some element of $\mathscr{A}$.*

(vii) *If $\{U_i\}_{i=1}^n$ is a finite collection of open sets then M can be chosen as $\{1, \cdots, n\}$ with $S_i \subset U_i$.*

Proof. See [[25], Theorem 16.3] and [[33], Theorem 2.13]. □

A sequence $\{\phi_i\}_{i\in M}$ satisfying conditions **(i)**−**(iv)** is **a partition of unity on** $\mathscr{A}$. If it satisfies **(v)**, it is said to be **of class** C_0^∞; if it satisfies **(vi)**, it is said to be **dominated by** $\mathscr{A}$ or **subordinated to** $\mathscr{A}$.
The previous theorem can be used to extend functions defined in domains with regular enough boundary. The following result gives us an important example of this situation in the Sobolev spaces.

Theorem 12. *Let $k \in \mathbb{N}$ and let Ω be a open set of class $C^{k-1,1}$ in $\mathbb{R}^N$. Then given $p \in [1,\infty)$ and any open set $\Omega' \supset \Omega$ there exists a linear bounded extension operator, i. e. there exists*

$$T : W^{k,p}(\Omega) \to W^{k,p}(\Omega')$$
$$u \to Tu,$$

such that $Tu = u$ a. e. in Ω.

Proof. See [[38], Theorem 7.25]. □

The following result is a "weak" version of the fundamental theorem of calculus. We provide a detailed proof for the lack of an accurate reference.

Theorem 13. *Let $p \in [1,\infty)$ and let $u \in W^{1,p}(\mathbb{R}^N)$. Given $h \in \mathbb{R}^N$ we have that*

$$u(x+h)-u(x)=\int_0^1 \nabla u(x+th)\cdot h dt, \qquad a.\ e. \quad x\in\mathbb{R}^N.$$

Proof. Let us fix $h \in \mathbb{R}^N$. In virtue of Friedrich's theorem (see [[2], Theorem 9.2]) there exists a sequence of $C_0^\infty(\mathbb{R}^N)$ functions $\{\phi_n\}_{n\in\mathbb{N}}$ converging to u in $W^{1,p}(\mathbb{R}^N)$. By the fundamental theorem of calculus each of these functions satisfies

$$\phi_n(x+h)-\phi_n(x)=\int_0^1 \nabla\phi_n(x+th)\cdot h dt, \tag{1.2}$$

for each $x \in \mathbb{R}^N$. It follows that (up to a subsequence), the left hand side of this expression converges to $u(x+h)-u(x)$ a. e. $x \in \mathbb{R}^N$.
In order to prove the convergence of the right hand side of (1.2) we apply Minkowski's integral inequality to get

$$\left|\int_{\mathbb{R}^N}\left|\int_0^1\Big(\nabla\phi_n(x+th)-\nabla u(x+th)\Big)\cdot h dt\right|^p dx\right|^{\frac{1}{p}} \le |h|\|\nabla\phi_n-u\|_{W^{1,p}(\mathbb{R}^N)}\to 0,$$

as $n\to\infty$. This shows that the function $x\mapsto \int_0^1 \nabla\phi_n(x+th)\cdot h dt$ converges to 0 in $L^p(\mathbb{R}^N)$ wich implies that, up to a subsequence,

$$\left\{\int_0^1 \nabla\phi_n(x+th)\cdot h dt\right\}_{n\in\mathbb{N}} \longrightarrow \int_0^1 \nabla u(x+th)\cdot h dt, \qquad \text{a. e. } x\in\mathbb{R}^N,$$

as $n\to\infty$. □

Theorem 14. *Let Ω and Ω' be open subsets of $\mathbb{R}^N$ and let $\Psi:\Omega\to\Omega'$ be a bi-Lipschitz function, i.e. a Lipschitz bijective function whose inverse is Lipschitz. If we set $J(x) := \Big|\det(Jac\Psi(x))\Big|$ then for any $g\in L^1(\Omega)$ we have that $g\circ\Psi^{-1}$ is a measurable function and that $(g\circ\Psi^{-1})\in L^1(\Omega')$; in this case:*

$$\int_\Omega g(y)J(y)dy=\int_{\Omega'} g\circ\Psi^{-1}(x)dx.$$

Proof. See [[14], Theorem 3.9] and [[16], Theorem 263D]. □

1.2 Fourier transform

In this section we introduce the Fourier transform and related concepts. Before giving the necessary definitions we present an important remark about the notation.

Remark 15. *Since Fourier analysis is developed using $\mathbb{C}$ as the underlying scalar field, it is necessary to introduce notation to differenciate some spaces of real-valued functions from their complex-valued counterpart.*
Hence, given $\Omega \subset \mathbb{R}^N$ and $p \in [1,\infty]$, we introduce the notation $\mathscr{M}(\Omega;\mathbb{C})$ for the measurable complex-valued functions defined on Ω, and $L^p(\Omega;\mathbb{C})$ for the space of measurable complex-valued functions with modulus p-integrable on Ω. We denote the norm of these spaces by $\|\cdot\|_{L^p(\Omega;\mathbb{C})}$.
In the same spirit we denote the Schwartz space of real-valued rapidly decreasing functions by $\mathscr{S}(\mathbb{R}^N)$ and its complex-valued counterpart by $\mathscr{S}(\mathbb{R}^N;\mathbb{C})$.

Note: All the statements proved for spaces of complex-valued functions also hold for their real-valued counterpart.

Definition 16. *Given $f \in L^1(\mathbb{R}^N;\mathbb{C})$ we define the Fourier transform of f by*

$$\mathscr{F}(f)(\xi) := \int_{\mathbb{R}^N} f(x)e^{-2\pi i \xi\cdot x}dx, \qquad \text{for } \xi \in \mathbb{R}^N. \tag{1.3}$$

Remark 17. *Throughout this work we will use exclusively the variable ξ as the argument of Fourier transforms of functions. On the other hand, in the cases when it is difficult to avoid writing the argument of the function to be transformed we will write explicitly the argument of the function inside the Fourier transform. For example, if the function $x \to |x|^k f(x)$ belongs to $L^1(\mathbb{R}^N;\mathbb{C})$ for some $k \in \mathbb{R}$, we will write its Fourier transform in the form $\mathscr{F}\left(|x|^k f(x)\right)$.*

Lemma 18. *The Fourier transform $\mathscr{F} : \mathscr{S}(\mathbb{R}^N;\mathbb{C}) \to \mathscr{S}(\mathbb{R}^N;\mathbb{C})$ is bicontinuous and bijective. Moreover, given $\psi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$ and any multi-index $\alpha \in \mathbb{N}_0^N$ we have*

(i) $\mathscr{F}(\partial^\alpha \psi)(\xi) = \xi^\alpha (2\pi i)^{|\alpha|}\mathscr{F}(\psi)(\xi)$.

(ii) $\partial^\alpha \mathscr{F}(\psi)(\xi) = (-2\pi i)^{|\alpha|}\mathscr{F}\left(x^\alpha \psi(x)\right)(\xi)$.

Proof. See [[17],Proposition 2.2.11]. □

2. Motivation

The aim of this work is to study certain type of operators that have adquired a renewed relevance in the last decade. This wide class of operators, genericaly called **nonlocal** operators, appears naturally in many applications in physics, probability, biology and economics, also they are relevent for themselves in mathematics because, as we will discusse below, many classical results involving differential operators can be understood as limiting cases of properties of these operators. Roughly speaking, we can understand a nonlocal operator as one operator defined in a functional space whose output or value depends on the whole domain of the input. In this work our attention is focused on a very important nonlocal operator known as the **fractional Laplacian**. The fractional Laplacian plays a similar role in the nonlocal operators as the one played by the Laplacian in the theory of ellpitic partial differential equations, i.e. the fractional Laplacian constitutes the most simple but important nonlocal operator and therefore it is considered as the benchmark in the study of these kind of operators.

2.1 Origin of the fractional Laplacian as an integro-differential diffusion operator

We start providing a rigorous definition of local operators to discuss how the nonlocal operators arise. For the sake of conciseness, in this section we will work with smooth functions defined in the whole space.

Definition 19. *A linear operator $L : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$ is called* ***local*** *if for every pair $f, g \in C^\infty(\mathbb{R}^N)$ and every $x \in \mathbb{R}^N$ we have that if f and g are equal in some neighborhood of x then $Lf(x) = Lg(x)$.*

It is clear that any linear differential operator satisfies this requirement. Moverover the converse is true, i.e. any local operator must be a linear differential operator, see [26].
In the study of elliptic differential equations other conditions are usually imposed to the local operators, most of these conditions are intended to guarantee some order preserving properties on the operators like the comparison or maximum principle. In its most basic form we will say that a linear operator satisfies the maximum principle if distinguishes the local maxima of any smooth function. This idea is summarized in the following definition.

Definition 20. *A linear operator $L : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$ satisfies the* ***maximum principle*** *if given $f \in C^\infty(\mathbb{R}^N)$ for every local maximizer x of f we have that $Lf(x) \le 0$.*

As all the local operators can be characterized as differential operators, the following theorem shows that every linear operator that satisfies the maximum principle is local and can be characterized in the following way.

The following result is well-known, nonetheless we provide a proof since we did not found an accurate reference for it.

Theorem 21. *Every linear operator $L : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$ that satisfies the maximum principle is a* ***diffusion operator****, i.e. has the form*

$$L = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i},$$

where a_{ij} and b_i are continuous and $(a_{ij}(x))_{1\le i,j\le N}$ is symmetric and positive semidefinite for every $x \in \mathbb{R}^N$.

Proof. Let $f \in C^\infty(\mathbb{R}^N)$. If we apply a second order Taylor's expansion around some fixed $x \in \mathbb{R}^N$ we get that

$$f(y) = f(x) + \nabla f(x)\cdot(y-x) + \frac{1}{2}(y-x)^t D^2 f(x)(y-x) + g(y),$$

with $g \in C^\infty(\mathbb{R}^N)$ satisfying that $\frac{g(y)}{\|y-x\|^2}$ goes to 0 as $y \to x$. The maximum principle directly implies that L vanishes at any constant function. If we define the functions $\beta_i(y) = y_i - x_i$, $\alpha_{ij}(y) = (y_i - x_i)(y_j - x_j)$, $b_i(y) = L(\beta_i)(y)$ and $a_{ij}(y) = L(\alpha_{ij})(y)$ then we can conclude by linearity that

$$L(f)(x) = \sum_{i=1}^{N} b_i(x)\frac{\partial f(x)}{\partial x_i} + \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + L(g)(x).$$

We claim that $Lg(x) = 0$. Indeed, given $\varepsilon > 0$ consider the function $y \mapsto g(y) + \varepsilon\|y-x\|^2$. By our hypoteses about g it follows that in a small enough punctured ball around x this function is strictly positive, hence, by the maximum principle and the linearity of L, we have that $L(g)(x) + \varepsilon L\left(\|y-x\|^2\right)(x) \ge 0$ for every $\varepsilon > 0$. This implies that $L(g)(x) \ge 0$. Changing ε by $-\varepsilon$ in the previous argument it follows that $L(g)(x) \le 0$.
Finally, in order to conclude the result it suffices to show that $(a_{ij}(x))_{1\le i,j\le N}$ is positive semidefinite, the symmetry follows directly from the fact that f is a smooth function. Hence, if we denote by $A(x)$ the matrix $(a_{ij}(x))_{1\le i,j\le N}$ we get that

$$\xi^t A(x)\xi = L\left(\sum_{i,j=1}^{N} \xi_j \alpha_{ij}\xi_i\right)(x),$$

which is nonnegative because the function $y \mapsto \sum_{i,j=1}^{N} \xi_j (y_j - x_j)(y_i - x_i)\xi_i = \|(y-x)^t\xi\|^2$ has x as a local minimizer. ☐

Notice that the previous theorem gives us inmediatly a very particular differential operator in spite of, apparently, dropping the assumption of being local. Roughly speaking, in this case, the condition of recognizing local maxima substitutes the hypothesis of being local in some sense. Let us consider the following nonlocal generalization of the maximum principle.

Definition 22. *A lineal operator $L : C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N)$ satisfies the* ***global maximum principle*** *if for every $f \in C_0^\infty(\mathbb{R}^N)$ we have that $Lf(x) \le 0$ whenever $x \in \mathbb{R}^N$ is a global maximizer of f.*

In this case we can obtain a result analogous to *Theorem* 21 adding the hipothesis of being local.

Theorem 23. *Let $L : C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N)$ be a linear operator. If L is local and satisfies the maximum global principle then there exist a continuous and nonnegative function $c : \mathbb{R}^N \to \mathbb{R}$ such that $L + c$ is a diffusion operator.*

Remark 24. *The previous theorem has an analogous when L is defined on $C^\infty(\mathbb{R}^N)$. In this case, the global maximum principle is changed by the so-called **positive global maximum principle**, which requires that the function is nonnegative at the point where the maximum is attained. This requirement of the nonnegativeness of the function is well-known in the theory of elliptic operators with **drift term**, i.e. the nonnegative function c of the previous theorem. Moreover, when the diffusion part is elliptic this result can be improved, see [[38], Theorem 3.5].*

In fact, the previous theorem is a particular case of the following theorem due to Courrege (see [9]) that characterizes the linear operators which satisfies the global maximum principle.

Theorem 25 (Courrege's theorem)**.** *A linear operator $L : C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N)$ satisfies the global maximum principle if and only if there exist a continuous nonnegative function $c : \mathbb{R}^N \to \mathbb{R}$, a diffusion operator $D : C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N)$ and a family of nonnegative measures $(d\mu_x)_{x\in\mathbb{R}^N}$ such that*

$$(L+c)u(x) = Du(x) + \int_{\mathbb{R}^N} (u(x+y) - u(x) - y \cdot \nabla u(x)\chi_{B_1}(y))d\mu_x(y), \tag{2.4}$$

with

$$\int_{\mathbb{R}^N} \min(y^2, 1)d\mu_x(y) < \infty. \tag{2.5}$$

Since L does not have to be necessarily local the second term of (2.4) appears. This integro-differential operator is usually known as the nonlocal component of L or the purely jumping part due to its interpretation in stochastic processes.
In this work we are interested in studying a very particular type of nonlocal operator component: those arising from assuming a quite simple family of measures $(d\mu_x)_{x\in\mathbb{R}^N}$ for the nonlocal part in the Courrege's theorem. Explicitly, we can simplify the nonlocal part assuming the following hypotheses.

1. **Absolutely continuity:** $d\mu_x(y) = K(x,y)dy$ for some function $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ that guarantees the condition (2.5) .

2. **Symmetry:** $K(x,-y) = K(x,y)$, $\forall x, y \in \mathbb{R}^N$.

3. **"Translational invariance":** $K(x,y) = K(x',y)$, $\forall x, x', y \in \mathbb{R}^N$.

Under the first two assumptions we can rewrite the nonlocal component of (2.4), using the symmetry of the term involving the gradient of the function and a suitable change of variable, as

$$Lu(x) = \frac{1}{2}\int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(x,y)dy.$$

If we add the term assumption we get that L has the previous form dropping the x dependence in K which implies that L commutes with translations.
It is customary to deal with the following two types of densities K.

1. **General stable:**
$$K(x,y) = \frac{a(x, y/|y|)}{|y|^{N+2s}}, \tag{2.6}$$
with $a \in L^1(\mathbb{S}^{N-1})$ and $s \in (0,1)$.

2. **Elliptic:** There exist $0 < \lambda \leq \Lambda$ and $s \in (0,1)$ such that
$$\frac{\lambda}{|y|^{N+2s}} \leq K(x,y) \leq \frac{\Lambda}{|y|^{N+2s}}. \tag{2.7}$$

2.2 Spectral theory and functional calculus

Analogously to the case of linear operators defined on $\mathbb{R}^N$, some bounded linear operators defined on Hilbert spaces can be decomposed in their "fundamental frequencies", i.e. in terms of their action on eigenvectors. It is well-know that the Laplacian operator (and other elliptic differential operators) also can be decomposed in that way in spite of not beign defined directly in a Hilbert space. There exist abstract ways to understand this decomposition like spectral theorems for densely defined unbounded linear operators, see [12]. Following this abstract approach, given a linear function $L : D \to H$, where D is a dense subset of a Hilbert space H, it can be defined the composition of L with certain type of functions whose domain contains the spectrum of L denoted by $\sigma(L)$. Explicitly, given a function $m : \sigma(L) \to \mathbb{C}$ we can define $m(L)$ acting through the spectrum of L. In the most simple case, when the spectrum of L is discrete and given by $(\mu_n)_{n\in\mathbb{N}}$ and the respective eigenvectors are given by $\{e_n\}_{n\in\mathbb{N}}$ we get that

$$m(L)(e_i) = m(\lambda_i)e_i, \qquad i \in \mathbb{N}.$$

This idea allows us to define heat operators e^{-tL}, resolvents $\frac{1}{L-z}$ for $z \notin \sigma(L)$ and, with significative relevance for this case, roots L^s with $s \in (0,1)$, whenever $\sigma(L) \subset \mathbb{R}_+$. In spite of the power of this method of the functional calculus it is preferred to follow a rather indirect approach that uses the classical spectral theorem [[1], Theorem 4.27]. This method consist in studying the linear boundary value problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The previous boundary value problem can be addressed using classical techniques of the functional analysis to obtain the inverse or solution operator of this problem. The solution operator is a well-defined linear, compact, self-adjoint and positive-definite operator, namely $T : L^2(\Omega) \to L^2(\Omega)$ which, in virtue of the splectral theorem [[1], Theorem 4.27] has a decomposition in eigenfunctions and which eigenvalues corresponds with the reciprocal of the eigenvalues of the Laplacian, [[2], Theorem 9.31]. Thus, as T has these properties it can be defined the operator T^s with $s \in (0,1)$ by means of the functional calculus which provides a definition for $(-\Delta)^s$, at least over the spectrum of the Laplacian.

2.3 The fractional Laplacian and the Fourier transform

As we discussed in the *Section* 2.2 , for some positive operators which are densely defined in a Hilbert space it is possible to define their fractional powers through the spectral theorem. These fractional powers can be thought as an intermediate step in the process of computing the original operator, for instance, as we commented in the section 2.2 we can decompose the positive operator

$$-\Delta = (-\Delta)^{1/2} \circ (-\Delta)^{1/2},$$

even though it seems not reasonable to compute the Laplacian of a function through computing twice the "half-Laplacian". This decomposition opens the question about the existence of functions that are not twice differentiable but regular enough to define their half-Laplacian. This discussion leads us directly to the problem of defining fractional derivatives. A classical way to do this is by means of the Fourier transform. Since the Fourier transform (roughly speaking) turns in a bijective way decay onto differentiability (see *Lemma* 18), we can change the condition about the fractional differentiability of a function by a fractional decay of its Fourier transform. This motivates the definition of the classical Sobolev spaces H^s.

Definition 26. *For $s > 0$ and $N \in \mathbb{N}$ we define*

$$H^s(\mathbb{R}^N) := \left\{ f \in L^2(\mathbb{R}^N) \quad \Big| \quad \int_{\mathbb{R}^N} (1+|\xi|^2)^s |\mathscr{F}(f)(\xi)|^2 d\xi < \infty \right\},$$

endowed with the inner product

$$\langle f, g\rangle_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (1+|\xi|^2)^s \mathscr{F}(f)(\xi)\overline{\mathscr{F}(g)(\xi)} d\xi, \quad f, g \in H^s(\mathbb{R}^N).$$

These spaces are defined looking for functions whose Fourier transform satisfy some decay property and, fortunately, they are quite nicely behaved too. Classical results (see [[18],Chapter 6]) prove that for any $s > 0$, $H^s(\mathbb{R}^N)$ is a complex Hilbert space having $\mathscr{S}(\mathbb{R}^N;\mathbb{C})$ as a dense subspace.

In order to translate the decay condition in the definition of $H^s(\mathbb{R}^N)$ into a differentiability condition, it is necessary to give the following definition.

Definition 27. *Given $s > 0$ and $f \in H^s(\mathbb{R}^N)$ we define*

$$D^s f(x) := \mathscr{F}^{-1}(|\xi|^s \mathscr{F}(f))(x), \tag{2.8}$$

and we call it the s-derivative (or homogeneous derivative of order s) of f.

Remark 28. *We note that for $s > 0$ and $f \in H^s(\mathbb{R}^N)$ we have that $|\xi|^s \mathscr{F}(f) \in L^2(\mathbb{R}^N;\mathbb{C})$, and then Plancherel's identity guarantees that $D^s f \in L^2(\mathbb{R}^N;\mathbb{C})$, particularly this guarantees the well-definition of $D^s f$.*

Remark 29. *Let us see that the homogeneous derivative does not coincide with the standard definition of the directional derivatives (not even in the one dimensional case).*
For instance, let us consider the function $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = e^{-\pi x^2} \in \mathscr{S}(\mathbb{R})$*. Clearly*

$$f'(x) = -2\pi x e^{-\pi x^2},$$

is an odd (nonzero) function.
On the other hand, since $\mathscr{F}(f) = f$ *we can compute its homogeneous derivative of order* 1 *as follows*

$$(D^1 f)(\xi) = \int_{\mathbb{R}} e^{-2\pi\xi x i}|x|e^{-\pi x^2}dx = \int_{\mathbb{R}} \cos(2\pi i\xi x)|x|e^{-\pi x^2}dx,$$

where the last equality comes from Euler's identity and the symmetry of the integrand. Noticing that $D^1(f)$ *is an even function whereas* f' *is odd we conclude that these two functions cannot coincide. We can go further and provide a power series representation for this homogeneous derivative emphasizing the differences with* f' *(whose power series representation is well-known)*

$$D^1 f(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}\pi^{n-1} n!}{(2n)!}\xi^n.$$

Even though this definition does not coincide with the standard definition of the directional derivatives, it gives a kind of total derivative in the following sense:

Proposition 30. *Let* $f \in L^2(\mathbb{R}^N;\mathbb{C})$*. Then for any* $n \in \mathbb{N}$ *we have that:*
$D^n f \in L^2(\mathbb{R}^N;\mathbb{C})$ *if and only if the weak derivatives of the real and and imaginary parts of* f *exist and* $\partial^\alpha Re(f), \partial^\alpha Im(f) \in L^2(\mathbb{R}^N)$ *for all* $|\alpha| \le n$*.*
In particular, $W^{n,2}(\mathbb{R}^N)$ *coincides with the subspace of real-valued functions in* $H^n(\mathbb{R}^N)$ *and, in this case, the norms* $\|\cdot\|_{H^n(\mathbb{R}^N)}$ *and* $\|\cdot\|_{W^{n,2}(\mathbb{R}^N)}$ *are equivalent when restricted to* $W^{n,2}(\mathbb{R}^N)$*.*
Moreover, the weak derivatives of a function $f \in W^{n,2}(\mathbb{R}^N)$ *are given by*

$$\partial^\alpha f = \mathscr{F}^{-1}\left(\xi^\alpha (2\pi i)^{|\alpha|}\mathscr{F}(f)\right). \tag{2.9}$$

Proof. See [[13],Section 5.8.4]. □

This idea can be easily generalized for a derivative of any order:

Proposition 31. *Given* $s > 0$ *and* $f \in H^s(\mathbb{R}^N)$ *the norms* $\|\cdot\|_{H^s(\mathbb{R}^N)}$ *and* $\|\cdot\|$ *defined by*

$$\|f\| := \|f\|_{L^2(\mathbb{R}^N;\mathbb{C})} + \|D^s f\|_{L^2(\mathbb{R}^N;\mathbb{C})}, \tag{2.10}$$

are equivalent. Indeed,

$$H^s(\mathbb{R}^N) = \left\{ f \in L^2(\mathbb{R}^N;\mathbb{C}) \quad \Big| \quad \hat{f}(\xi)|\xi|^s \in L^2(\mathbb{R}^N;\mathbb{C})\right\}. \tag{2.11}$$

Remark 32. *Beyond the last considerations we can show that, for some particular values of s, the operators* D^s *come, in fact, from derivation processes. For example, given* $u \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$ *we can apply Lemma 18 to obtain:*

$$\mathscr{F}(\Delta u)(\xi) = \sum_{i=1}^{n} -4\pi^2\xi_i^2 \mathscr{F}(u)(\xi) = -4\pi^2|\xi|^2\mathscr{F}(u), \tag{2.12}$$

showing that $-\Delta u = (2\pi)^2 D^2 u$. *Hence, (up to constants) the differential operators of order* s *in the case* $s \in (0,2)$ *give us a new approximation to the fractional Laplacian. We can illustrate this idea with the case* $s = 1$,

$$D^1 D^1 u = \mathscr{F}^{-1}(|\xi|\mathscr{F}(D^1 u)) = \mathscr{F}^{-1}(|\xi|^2 \mathscr{F}(u)) = \frac{1}{4\pi^2}(-\Delta u).$$

This discussion motivates an alternative definition for the fractional Laplacian.

Definition 33. *For* $s \in (0,1)$ *and* $u \in H^{2s}(\mathbb{R}^N)$ *we define the fractional Laplacian of* u *order* s *(or* s*-fractional Laplacian) by*

$$(-\Delta)^s u := (2\pi)^{2s} D^{2s}(u). \tag{2.13}$$

Remark 34. *The constant* $(2\pi)^{2s}$ *is only relevant for studies about the asymptotic behaviour of the operator* $(-\Delta)^s$ *when* s *approaches* 0 *or* 1. *For example when* $s = 0$ *it is clear that:*

$$D^0 u = u, \qquad \forall u \in \mathscr{S}(\mathbb{R}^N;\mathbb{C}),$$

and when $s = 1$ *we have that*

$$D^2 u = \frac{1}{4\pi^2}\Delta u, \qquad \forall u \in \mathscr{S}(\mathbb{R}^N;\mathbb{C}).$$

This constant appears due to our particular choosing of the normalizing constant in the Fourier transform . In the case when the Fourier transform is defined with the normalization term outiside the integral this constant is omitted (see for instance [11]). However, there exist more suitable normalizing constants to study the asymptotic properties of the fractional Laplacian. We will discuss later how should be selected the constant for these purposes.

Remark 35. *With the new notation introduced we can characterize the Sobolev spaces* $H^{2s}(\mathbb{R}^N)$ *for* $s \in (0,1)$ *in terms of the fractional Laplacian using Proposition 31, namely*

$$H^{2s}(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N;\mathbb{C})| \quad \middle| \quad (-\Delta)^s u \in L^2(\mathbb{R}^N;\mathbb{C}) \right\},$$

endowed with the norm

$$\|u\| := \|u\|_{L^2(\mathbb{R}^N;\mathbb{C})} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N;\mathbb{C})}.$$

This remark points out an important analogy between the s-fractional Laplacian and the Laplacian. These two operators characterize the Sobolev spaces with the least regularity where they make pointwise sense (almost everywhere), namely $H^{2s}(\mathbb{R}^N)$ and $W^{2,2}(\mathbb{R}^N)$, respectively. Nonetheless, in spite of this characterization, it is usual to look for solutions of equations involving these operators in spaces with less regularity.

For example, given $f \in L^2(\mathbb{R}^N)$ we can go from the strong formulation for the Poisson equation,

$$-\Delta u(x) = f(x), \qquad \text{a. e. } x \in \mathbb{R}^N, \tag{2.14}$$

to the weak variational formulation

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v = \int_{\mathbb{R}^N} f v, \qquad \forall v \in W^{1,2}(\mathbb{R}^N), \tag{2.15}$$

using standard arguments of integration by parts and density.
Analogously, given $s \in (0,1)$ and $f \in L^2(\mathbb{R}^N;\mathbb{C})$ we can go from the strong formulation for the Poisson equation for the s-Laplacian,

$$-(\Delta)^s u(x) = f(x), \qquad \text{a. e. } x \in \mathbb{R}^N, \tag{2.16}$$

to the weak variational formulation

$$(2\pi)^{2s} \int_{\mathbb{R}^N} |\xi|^{2s} \mathscr{F}(u)(\xi)\overline{\mathscr{F}(v)}(\xi) d\xi = \int_{\mathbb{R}^N} f\overline{v}, \qquad \forall v \in H^s(\mathbb{R}^N), \tag{2.17}$$

multiplying on both sides by $\overline{v}$ with $v \in H^s(\mathbb{R}^N)$ and using standard properties of the Fourier transform.
Again, keeping track of the analogies between these two operators, we notice that the weak variational formulations reduce the regularity requirements in the solutions "to the half" on both cases, passing from $W^{2,2}(\mathbb{R}^N)$ in (2.14) to $W^{1,2}(\mathbb{R}^N)$ in (2.15) and passing from $H^{2s}(\mathbb{R}^N)$ in (2.16) to $H^s(\mathbb{R}^N)$ in (2.17), respectively.
Another important analogy between these formulations is that (2.17) also has variational form restricted to the subspace of real-valued functions in $H^s(\mathbb{R}^N)$. To prove this last claim it is necessary to show that our formulations (2.16) and (2.17) represent equations with underlying scalar field given by $\mathbb{R}$. To this end we provide, in the following two chapters, alternative definitions of $H^s(\mathbb{R}^N)$ and of $(-\Delta)^s$ that do not depend (explicitly) on the Fourier transform.

3. Fractional Sobolev Spaces

3.1 Motivation

For the cases $s = n \in \mathbb{N}$ we have a direct interpretation of the Sobolev spaces $H^s(\mathbb{R}^N)$ in terms of (weak) differentiability conditions given by *Proposition* 30. This equivalent formulation is useful because, in general, the Fourier transform does not preserve neither the compactness of the support of a function nor the localization of it and, on the other hand, the weak differentiability allows us to localize the function. Also, since the idea of weak derivative can be defined in any open set, the equivalent definition given by the spaces $W^{n,2}$ provides a suitable functional framework to extend the definition of H^n to other domains different from $\mathbb{R}^N$. This discussion suggests that, in order to generalize the definition of H^s for any $s > 0$ to more general domains, it is necessary to give a characterization of the space $H^s(\mathbb{R}^N)$ without using the Fourier transform. Let us start introducing some notation.

Definition 36. *Fpr $s \in (0,1)$ and $u \in \mathcal{M}(\mathbb{R}^N;\mathbb{C})$ we define the Gagliardo's seminorm of u by*

$$[u]^{s,2}_{\mathbb{R}^N} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}. \tag{3.18}$$

The following theorem gives us the desired characterization of the space $H^s(\mathbb{R}^N)$ in the case $s \in (0,1)$.

Theorem 37. *For $s \in (0,1)$,*

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N;\mathbb{C}) \quad | \quad [u]^{s,2}_{\mathbb{R}^N} < \infty\}, \tag{3.19}$$

Moreover, for $u \in H^s(\mathbb{R}^N)$ we have that

$$\|D^s u\|_{L^2\mathbb{C}(\mathbb{R}^N)} = C[u]^{s,2}_{\mathbb{R}^N}, \tag{3.20}$$

for some positive constant C only depending on N and s.

Proof. Given $u \in H^s(\mathbb{R}^N)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy &= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+2s}}\int_{\mathbb{R}^N} |u(z+y)-u(y)|^2 dydz \\
\text{(Plancherel's identity)} \qquad &= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+2s}}\int_{\mathbb{R}^N} |e^{-2\pi i z\cdot\xi}\mathcal{F}(u)(\xi)-\mathcal{F}(u)(\xi)|^2 d\xi dz \\
&= \int_{\mathbb{R}^N} |\mathcal{F}(u)(\xi)|^2 \int_{\mathbb{R}^N} \frac{|e^{-2\pi i z\cdot\xi}-1|^2}{|z|^{N+2s}} dzd\xi \\
&= \int_{\mathbb{R}^N} |\mathcal{F}(u)(\xi)|^2 \int_{\mathbb{R}^N} \frac{4\sin^2(\pi\xi\cdot z)}{|z|^{N+2s}} dzd\xi.
\end{aligned} \tag{3.21}$$

In order to estimate this last integral it is necessary to study the term associated to the inner integral, namely

$$\int_{\mathbb{R}^N} \frac{4\sin^2(\pi\xi\cdot z)}{|z|^{N+2s}} dzd\xi. \tag{3.22}$$

Analyzing this integral we will arrive to the expression

$$\int_{\mathbb{R}^N} \frac{4\sin^2(\pi y_1)}{|y|^{N+2s}} dy, \tag{3.23}$$

which does not depends on ξ. The integrability of (3.23) is guaranteed, near the origin, by the inequality $\sin^2(\pi y_1) \le \pi^2 |y|^2$ for all $y \in \mathbb{R}$ and its integrability, far from the origin, is provided by the boundedness of $\sin^2(\pi y_1)$.
Returning to (3.22) and considering the change of variables $z = \frac{y}{|\xi|}$ we get

$$\int_{\mathbb{R}^N} \frac{4\sin^2(\pi \xi \cdot z)}{|z|^{N+2s}} dz d\xi = |\xi|^{2s} \int_{\mathbb{R}^N} \frac{4\sin^2(\pi \frac{\xi}{|\xi|} \cdot y)}{|y|^{N+2s}} dy.$$

Applying a change of variables given by an orthogonal matrix A such that $Ae_1 = \frac{\xi}{|\xi|}$, it follows that

$$\int_{\mathbb{R}^N} \frac{4\sin^2(\pi \frac{\xi}{|\xi|} \cdot y)}{|y|^{N+2s}} dy = \int_{\mathbb{R}^N} \frac{4\sin^2(\pi y_1)}{|y|^{N+2s}} dy,$$

where y_1 is the first component of the vector y.
Finally, these last considerations and (3.21) implies

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy = C \int_{\mathbb{R}^N} |\mathscr{F}(u)(\xi)|^2 |\xi|^{2s} d\xi,$$

with C given by (3.23). □

Remark 38. *The condition imposed by the convergence of* (3.18) *can be analyzed in two parts. First, the convergence at the infinity. This condition automatically holds for functions in $L^2(\mathbb{R}^N;\mathbb{C})$. Secondly, the convergence near the origin. This condtion is, in some sense, a condition about the regularity of the functions in $H^s(\mathbb{R}^N)$. In order to clarify this last part let us consider a function $u \in L^2(\mathbb{R}^N;\mathbb{C})$ such that for some $\gamma \in (0,1]$, some fixed $\delta > 0$ for and some positive constant $c > 0$ satisfies $|u(x) - u(y)| \le c|x - y|^{\gamma}$ whenever $|x - y| < \delta$.*
In this particular case we have that

$$\int_{B_1(0)} \int_{|x-y|<\delta} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \le c^2 \int_{B_1(0)} \int_{|x-y|<\delta} \frac{1}{|x - y|^{N+2(s-\gamma)}} dx dy.$$

This estimation show us that we can guarantee the convergence of (3.18) *near the origin for "locally uniform" γ-Hölder continuous functions with $\gamma > s$. Thus, roughly speaking, we can think the convergence condition on* (3.18) *as a weak analogous of γ-Hölder continuity.*

Remark 39. *Given $s \in (0,1)$ and $u \in H^s(\mathbb{R}^N)$, we can use Theorem 37 to change the non local condition stated in terms of complex-valued functions imposed in the definition of $H^s(\mathbb{R}^N)$, namely*

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\mathscr{F}(u)(\xi)|^2 d\xi < \infty,$$

by the new condition

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy<\infty,$$

that does not involve any expression in the complex numbers and that can be localized, in some sense, as we pointed out in the Remark 38.

3.2 Definitions and basic properties

The previous discussion motivates the general definition of the fractional Sobolev spaces of real-valued functions.

Definition 40. *Let Ω be an open subset of $\mathbb{R}^N$ and let $s\in(0,1)$, $k\in\mathbb{N}_0$ and $1\le p<\infty$. The fractional Sobolev space $W^{k+s,p}(\Omega)$ is defined by*

$$W^{k+s,p}(\Omega):=\left\{u\in W^{k,p}(\Omega)\ \middle|\ [\partial^\alpha u]_\Omega^{s,p}:=\left(\int_\Omega\int_\Omega\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}<\infty,\quad\forall\alpha\in\mathbb{N}_0^N,|\alpha|=k\right\},$$

endowed with the norm

$$\|u\|_{W^{k+s,p}(\Omega)}:=\left(\|u\|^p_{W^{k,p}(\Omega)}+\sum_{|\alpha|=k}\left([\partial^\alpha u]_\Omega^{s,p}\right)^p\right)^{\frac{1}{p}}.\tag{3.24}$$

For the case $p=\infty$

$$W^{k+s,\infty}(\Omega):=\left\{u\in W^{k,\infty}(\Omega)\ \middle|\ [\partial^\alpha u]_\Omega^{s,\infty}:=\operatorname*{ess\,sup}_{\substack{x,y\in\Omega\\x\neq y}}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|}{|x-y|^s}<\infty,\quad\forall\alpha\in\mathbb{N}_0^N,|\alpha|=k\right\},$$

with its respective norm given by

$$\|u\|_{W^{k+s,\infty}(\Omega)}:=\max\left\{\|u\|_{W^{k,\infty}(\Omega)},\max_{|\alpha|=k}\operatorname*{ess\,sup}_{\substack{x,y\in\Omega\\x\neq y}}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|}{|x-y|^s}\right\}.\tag{3.25}$$

The following definition unifies the notation of the integer Sobolev spaces introduced in the *Definition* 8 with the fractional Sobolev spaces introduced in the last definition.

Definition 41. *Given $r\in\mathbb{R}_+$, Ω an open subset of $\mathbb{R}^N$ and $p\in[1,\infty]$ we define*

$$W^{r,p}(\Omega):=W^{\lfloor r\rfloor+frac(r),p}(\Omega),\tag{3.26}$$

where $\lfloor r\rfloor$ represents the integer part of r and $frac(r):=r-\lfloor r\rfloor$ represents the fractional part of r. When $r\in\mathbb{N}_0$ the right hand side of (3.26) is understood as in Definition 8, in other case we understand it as in Definition 40.

The following proposition generalizes *Proposition* 30 for any real number r. Since it is not clearly enunciated, up to the best of our knowledge, in any reference we provide a proof.

Proposition 42. *Given $r \in \mathbb{R}_+$, $W^{r,2}(\mathbb{R}^N)$ coincides with the subspace of real-valued functions in $H^r(\mathbb{R}^N)$.*
Moreover, the norms $\|\cdot\|_{W^{r,2}(\mathbb{R}^N)}$ and $\|\cdot\|_{H^r(\mathbb{R}^N)}$ restricted to $W^{r,2}(\mathbb{R}^N)$ are equivalent.

Proof. Clearly, the case $r \in \mathbb{N}$ follows from *Proposition* 30.
Otherwise, if we set $k = \lfloor r \rfloor$ and $s = \mathrm{frac}(r)$, *Proposition* 31 implies that

$$H^r(\mathbb{R}^N) = \left\{ f \in H^k(\mathbb{R}^N) \quad \middle| \, |\xi|^k \mathscr{F}(f)(\xi) \in H^s(\mathbb{R}^N) \right\},$$

hence *Lemma* 18 implies that a function $f \in L^2(\mathbb{R}^N)$ belongs to $H^r(\mathbb{R}^N)$ if and only if $\partial^\alpha f \in H^s(\mathbb{R}^N)$ for each $|\alpha| = k$. Combining this fact with *Theorem* 37 we obtain the proof of the first part of the proposition.
The equivalence of the norms follows from *Theorem* 37, *Lemma* 18 and the following inequalities

$$|\xi^\alpha| \le (1+|\xi|^2)^{\frac{r}{2}}, \qquad \forall \xi \in \mathbb{R}^N, \quad \forall |\alpha| \le k,$$

and

$$|\xi^\alpha||\xi|^s \le (1+|\xi|^2)^{\frac{r}{2}} \le C\left(\sum_{|\beta|\le k} |\xi^\beta| + \sum_{|\beta|=k} |\xi^\beta||\xi|^s \right), \qquad \forall \xi \in \mathbb{R}^K, \quad \forall |\alpha| = k,$$

for some $C > 0$ depending only on N.

□

The following proposition summarizes some important properties of these spaces. The proof proposition generalize the ideas used in the proof of [[2], Proposition 9.1].

Proposition 43. *Let Ω be an open subset of $\mathbb{R}^N$, $s \in (0,1)$ and $k \in \mathbb{N}_0$. Then:*

(i) *For $1 \le p \le \infty$, $W^{k+s,p}(\Omega)$ is a Banach space endowed with the norms defined in* (3.24) *and* (3.25).

(ii) *For $1 < p < \infty$, $W^{k+s,p}(\Omega)$ is reflexive.*

(iii) *For $1 \le p < \infty$, $W^{k+s,p}(\Omega)$ is separable.*

Proof. (i) Given $k \in \mathbb{N}_0$ let us define the numbers

$$k^* := \left| \left\{ \alpha \in (\mathbb{N}_0)^N \,\middle|\, |\alpha| \le k \right\} \right|, \qquad \hat{k} := \left| \left\{ \alpha \in (\mathbb{N}_0)^N \,\middle|\, |\alpha| = k \right\} \right|$$

where the bars denote the number of elements of a set. Using this notation let us define:

$$J : W^{k+s,p}(\Omega) \to \left[L^p(\Omega)\right]^{k^*} \times \left[L^p(\Omega \times \Omega)\right]^{\hat{k}}$$
$$u \to \left((\partial^\alpha u)_{|\alpha|\le k}, \left(\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{\frac{N}{p}+s}} \right)_{|\alpha|=k} \right),$$

where the product space $[L^p(\Omega)]^{k^*} \times [L^p(\Omega\times\Omega)]^{\hat{k}}$ is endowed with the finite dimensional p-norm. Clearly, by construction J is a linear isometry onto its image. Thus, it suffices to show that $J\left(W^{k+s,p}(\Omega)\right)$ is closed. To this end let us consider a sequence $\{J(u_n)\}_{n\in\mathbb{N}}$ in $J\left(W^{k+s,p}(\Omega)\right)$ that satisfies

$$J(u_n) \to \Psi, \quad n \to \infty, \qquad \text{for some } \Psi \in \left[L^p(\Omega)\right]^{k^*} \times \left[L^p(\Omega \times \Omega)\right]^{\hat{k}}.$$

Arguing in the first k^* components of J, since the space $W^{k,p}(\Omega)$ is complete (see [2]) there exists $u \in W^{k,p}(\mathbb{R}^N)$ such that the first k^* components of Ψ are given by $(\partial^\alpha u)_{|\alpha|\le k}$. On the other hand, by the convergence of the last $\hat{k}$ components, given any multi-index α, with $|\alpha| = k$, there exists $f_\alpha \in L^p(\Omega \times \Omega)$ such that

$$\frac{|\partial^\alpha u_n(x) - \partial^\alpha u_n(y)|}{|x-y|^{\frac{N}{p}+s}} \to f_\alpha, \quad \text{a.e.} \quad (x,y) \in \Omega \times \Omega \qquad \text{(up to a subsequence).}$$

Besides, since $\partial^\alpha u_n(x) \to \partial^\alpha u(x)$ a.e. in Ω (up to the same subsequence) we get that

$$\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{\frac{N}{p}+s}} = f_\alpha(x,y), \quad \text{a.e.} \quad (x,y) \in \Omega \times \Omega.$$

This proves the closedness of the image of J and therefore the completeness of $W^{k+s,p}(\Omega)$.

(ii) Since $[L^p(\Omega)]^{k^*} \times [L^p(\Omega \times \Omega)]^{\hat{k}}$ is reflexive for $1 < p < \infty$ we can use the isometry J and the fact that $J\left(W^{k+s,p}(\Omega)\right)$ is closed to conclude the result.

(iii) Analogously to the last case, the separability of $W^{k+s,p}(\Omega)$ follows from the separability of $[L^p(\Omega)]^{k^*} \times [L^p(\Omega \times \Omega)]^{\hat{k}}$ for $1 \le p < \infty$, since any subset of a separable metric space is also a separable metric space. Finally, $W^{k+s,p}(\Omega)$ is separable since it is isometrically isomorphic to a separable space.

□

Remark 44. *Given an open set Ω in $\mathbb{R}^N$, $s \in (0,1)$ and $k \in \mathbb{N}_0$, the space $W^{k+s,2}(\Omega)$ is a real Hilbert space endowed with the inner product:*

$$\langle u,v\rangle_{W^{k+s,2}(\Omega)} := \langle u,v\rangle_{W^{k,2}(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \left(\frac{(\partial^\alpha u(x) - \partial^\alpha u(y))}{|x-y|^{\frac{N}{2}+s}} \right) \left(\frac{\partial^\alpha v(x) - \partial^\alpha v(y)}{|x-y|^{\frac{N}{2}+s}} \right) dxdy, \quad (3.27)$$

for any $u,v \in W^{k+s,2}(\Omega)$.

As expected, the regularity of the functions in these Sobolev spaces increases as the index of the space increases, at least in the case $\Omega = \mathbb{R}^N$. However, in order to conclude that weak differentiable functions (in the $W^{k,p}$ sense) have integrable fractional derivatives, it is necessary to impose some regularity conditions on their domain. The first two parts of the following proof are taken from [11] and the last part is a contribution of this work.

Theorem 45. *Let Ω be an open subset of $\mathbb{R}^N$ and let $0 < s' < s < 1$. Then*

(i) *$W^{s,p}(\Omega) \subset W^{s',p}(\Omega)$ for $1 \le p \le \infty$.*

(ii) *If Ω is a open set of class $C^{0,1}$, for $1 \le p < \infty$, we have that $W^{1,p}(\Omega) \subset W^{s,p}(\Omega)$.*

(iii) *If Ω is convex we have that $W^{1,\infty}(\Omega) \subset W^{s,\infty}(\Omega)$.*

Even more, in all the cases the inclusions are continuous.

Proof. (i) For the case $p \in [1,\infty)$, given $u \in W^{s,p}(\Omega)$ let us consider

$$\left([u]_{\Omega}^{s',p}\right)^p = \underbrace{\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps'}}\chi_{\{(x,y)||x-y|<1\}}dxdy}_{I_1(s',p)} + \underbrace{\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps'}}\chi_{\{(x,y)||x-y|\ge 1\}}dxdy}_{I_2(s',p)}. \tag{3.28}$$

Clearly, the first term is bounded by

$$I_1(s',p) \le \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\chi_{\{(x,y)||x-y|<1\}}dxdy.$$

On the other hand, for I_2 we have that

$$\begin{aligned} I_2(s',p) \le & 2^p\int_\Omega\int_\Omega \frac{|u(x)|^p+|u(y)|^p}{|x-y|^{N+ps'}}\chi_{\{(x,y)||x-y|\ge 1\}}dxdy \\ =& 2^{p+1}\int_\Omega\int_\Omega \frac{|u(x)|^p}{|x-y|^{N+ps'}}\chi_{\{(x,y)||x-y|\ge 1\}}dxdy \\ \le & 2^{p+1}\int_\Omega |u(x)|^p\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+ps'}}\chi_{\{(x,y)||x-y|\ge 1\}}dydx \\ =& C\|u\|^p_{L^p(\Omega)}, \end{aligned} \tag{3.29}$$

where

$$C = 2^{p+1}\int_{B_1(0)^c}\frac{1}{|z|^{N+ps'}}dy < \infty.$$

From the estimates for $I_1(s',p)$ and $I_2(s',p)$ it follows that there exists a constant C such that

$$\left([u]_{\Omega}^{s',p}\right)^p \le \left([u]_{\Omega}^{s,p}\right)^p + C\|u\|^p_{L^p(\Omega)}.$$

For the case $p=\infty$ we can use a similar decomposition given by:

$$[u]_{\Omega}^{s',\infty} = \max\Bigg\{\underbrace{\operatorname*{ess\,sup}_{x,y\in\Omega,0<|x-y|<1}\frac{|u(x)-u(y)|}{|x-y|^{s'}}}_{I_1(s',\infty)}, \underbrace{\operatorname*{ess\,sup}_{x,y\in\Omega,|x-y|\ge 1}\frac{|u(x)-u(y)|}{|x-y|^{s'}}}_{I_2(s',\infty)}\Bigg\}. \tag{3.30}$$

Analogously to the last case the first term of this decomposition is easily bounded by

$$I_1(s',\infty) \le \operatorname*{ess\,sup}_{x,y\in\Omega,0<|x-y|<1}\frac{|u(x)-u(y)|}{|x-y|^{s}}.$$

On the other hand, for $I_2(s',\infty)$ the bound is directly given by

$$I_2 \leq 2\|u\|_{L^\infty(\Omega)},$$

getting a bound of the form

$$[u]_\Omega^{s',\infty} \leq \max\left\{[u]_\Omega^{s,\infty}, C\|u\|_{L^\infty(\Omega)}\right\}.$$

(ii) Let $u \in W^{1,p}(\Omega)$. Let us consider again:

$$\left([u]_\Omega^{s,p}\right)^p = I_1(s,p) + I_2(s,p),$$

where $I_1(s,p)+I_2(s,p)$ have the same meaning as in (3.28). Since the bound for $I_2(s,p)$ still holds in this case, it suffices to analyze the first term. By the regularity of Ω, *Theorem* 12 guarantees the existence of an extension of u given by $Tu \in W^{1,p}(\mathbb{R}^N)$ such that $\|Tu\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for some positive constant $C > 0$. Hence, this considerations implies

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\chi_{\{(x,y)||x-y|<1\}}dxdy &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Tu(x)-Tu(y)|^p}{|x-y|^{N+ps}}\chi_{\{(x,y)||x-y|<1\}}dxdy \\
(z = x-y) \qquad &\leq \int_{\mathbb{R}^N}\int_{B_1(0)} \frac{|Tu(z+y)-Tu(y)|^p}{|z|^{N+ps}}dzdy \\
(\textit{Theorem } 13) \qquad &\leq \int_{\mathbb{R}^N}\int_{B_1(0)} \frac{|\int_0^1 \nabla Tu(y+tz)|^p dt}{|z|^{N-p(1-s)}}dzdy \\
(\text{Jensen's inequality}) \qquad &\leq \int_0^1\int_{B_1(0)} \frac{\|Tu\|^p_{W^{1,p}(\mathbb{R}^N)}}{|z|^{N-p(1-s)}}dzdt \\
&\leq C\|u\|^p_{W^{1,p}(\Omega)},
\end{aligned}$$

where in the second inequality we used the change of variables $z = x-y$.

(iii) Let $u \in W^{1,\infty}(\Omega)$. Let us consider again

$$[u]_\Omega^{s,\infty} = \max\{I_1(s,\infty), I_2(s,\infty)\}.$$

Arguing as in the first part, it follows that $I_2(s,\infty)$ is bounded and $I_2(s,\infty) \leq 2\|u\|_{L^\infty(\Omega)}$. Then, it suffices to estimate the first term. Since Ω is convex, [[2], Proposition 9.3] implies that u has a continuous representative in Ω and this representative satisfies the mean value inequality

$$|u(x)-u(y)| \leq \|\nabla u\|_{L^\infty(\Omega)}|x-y|, \qquad \forall x,y \in \Omega,$$

therefore

$$I_1 \leq \|\nabla u\|_{L^\infty(\Omega)}.$$

Finally,

$$[u]_\Omega^{s,\infty} \leq \max\left\{\|\nabla u\|_{L^\infty(\Omega)}, 2\|u\|_{L^\infty(\Omega)}\right\}.$$

□

Remark 46. *For an arbitrary open set Ω, [[2], Proposition 9.3] guarantees that for any $u \in W^{1,\infty}(\Omega)$ its continuous representative satisfies*

$$|u(x)-u(y)| \leq \|\nabla u\|_{L^\infty(\Omega)} dist(x,y), \qquad \forall x,y \in \Omega,$$

where $dist(x,y)$ is the geodesic distance between x and y in Ω. Therefore, to prove the inclusion $W^{1,\infty}(\Omega) \subset W^{s,\infty}(\Omega)$, it is enough to suppose that Ω satisfies that

$$\frac{dist(x,y)}{|x-y|^s} \leq K,$$

for some constant $K > 0$ and for all $x,y \in \Omega$ with $0 < |x-y| < 1$. Therefore, (roughly speaking) the condition imposed for this inclusion is a condition on the inner regularity of the domain Ω, whereas the condition imposed for the inclusion presented in the second item of Theorem 45 is about the regularity on the boundary of Ω.

Corollary 47. *Let $s',s \in (0,1)$, $k,k' \in \mathbb{N}_0$ be such that $k+s > k'+s'$ and let Ω be an open subset of $\mathbb{R}^N$. If $p \in [1,\infty)$ $(p=\infty)$ and Ω is of class $C^{0,1}$ (respectively convex), then the inclusion*

$$W^{k+s,p}(\Omega) \subset W^{k'+s',p}(\Omega)$$

is continuous.

Proof. For any value of p, if $k = k'$ then the first item of *Theorem* 45 guarantees the result. In other case, if $k > k'+s'$ and $p < \infty$ then together the trivial inclusion $W^{k+s,p}(\Omega) \subset W^{k,p}(\Omega)$ and the second part of *Theorem* 45 implies the result. If $p = \infty$ the result follows from the third part of *Theorem* 45. □

We present now the analogous continuous inclusions when the integrability parameter p varies.

Theorem 48. *Let $0 < s' < s < 1$, $1 \leq p' < p \leq \infty$ and let Ω be an open subset of $\mathbb{R}^N$ such that $|\Omega| < \infty$. Then:*

$$W^{s,p}(\Omega) \subset W^{s',p'}(\Omega),$$

in a continuous way.

Proof. Since Ω has finite measure, Hölder inequality implies that for $1 \leq p' < p \leq \infty$, $\|u\|_{L^{p'}(\Omega)} \leq C\|u\|_{L^p(\Omega)}$.

On the other hand, let us consider again the decomposition

$$\left([u]_\Omega^{s',p'}\right)^{p'} = \underbrace{\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p'}}{|x-y|^{N+p's'}} \chi_{\{(x,y)||x-y|<1\}} dxdy}_{I_1(s',p')} + \underbrace{\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p'}}{|x-y|^{N+p's'}} \chi_{\{(x,y)||x-y|\geq 1\}} dxdy}_{I_2(s',p')}.$$

The second term $I_2(s',p')$ is estimated exactly as in (3.29) implying that

$$I_2(s',p') \le C\|u\|^{p'}_{L^{p'}(\Omega)} \le C\|u\|^{p'}_{L^p(\Omega)}.$$

To estimate the first term $(I_1(s',p'))$ it is necessary to consider two cases.
If p is finite, we can apply Hölder inequality with the conjugate exponents $\frac{p}{p'}$ and $\frac{p}{p-p'}$ to get

$$\begin{aligned} I_1(s',p') =& \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p'}}{|x-y|^{N\frac{p'}{p}+sp'}} \frac{1}{|x-y|^{N\frac{p-p'}{p}-p'(s-s')}} \chi_{\{(x,y)||x-y|<1\}} dxdy \\ \le& C\left([u]^{s,p}_\Omega\right)^{p'}, \end{aligned}$$

where

$$C = \left(\int_\Omega\int_\Omega \frac{1}{|x-y|^{N-pp'\frac{s-s'}{p-p'}}} \chi_{\{(x,y)||x-y|<1\}} dxdy\right)^{\frac{p-p'}{p}},$$

is finite because the first integral involves an integrand compactly supported around y, $N-pp'\frac{s-s'}{p-p'} < N$ and Ω has finite measure.

If $p=\infty$ it follows that

$$\begin{aligned} I_1(s',p') =& \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p'}}{|x-y|^{sp'}} \frac{1}{|x-y|^{N-p'(s-s')}} \chi_{\{(x,y)||x-y|<1\}} dxdy \\ \le& C\left([u]^{s,\infty}_\Omega\right)^{p'} dxdy, \end{aligned}$$

where

$$C = \int_\Omega\int_\Omega \frac{1}{|x-y|^{N-p'(s-s')}} \chi_{\{(x,y)||x-y|<1\}} dxdy < \infty,$$

again because $N-p'(s-s') < N$ and Ω has finite measure. □

The following sequence of propositions provides useful results that will facilitate some computations in these spaces.

Proposition 49. *Let Ω be an open subset of $\mathbb{R}^N$, let $0<s<\gamma\le 1$ and let $1\le p\le\infty$. Then given $u\in W^{s,p}(\Omega)$ and $\phi\in C^{0,\gamma}_B(\overline{\Omega})$, we have that $u\phi\in W^{s,p}(\Omega)$, and*

$$\|u\phi\|_{W^{s,p}(\Omega)} \le C\|\phi\|_{C^{0,\gamma}_B(\overline{\Omega})}\|u\|_{W^{s,p}(\Omega)}. \tag{3.31}$$

Proof. First of all, let us notice that

$$\|u\phi\|_{L^p(\Omega)} \le \|\phi\|_{C^{0,\gamma}_B(\overline{\Omega})}\|u\|_{L^p(\Omega)},$$

for any $1\le p\le\infty$.

Assume $p < \infty$ and consider the decomposition proposed in (3.28)

$$\left([u\phi]_{\Omega}^{s,p}\right)^p = I_1(u\phi, s, p) + I_2(u\phi, s, p).$$

Since $u\phi \in L^p(\Omega)$ it follows that:

$$I_2(u\phi, s, p) \le C\|u\phi\|_{L^p(\Omega)} \le C\|\phi\|_{C_B^{0,\gamma}(\overline{\Omega})}\|u\|_{L^p(\Omega)}.$$

With respect to the term $I_1(u\phi, s, p)$ we have

$$\begin{aligned}
I_1(u\phi, s, p) =& \int_\Omega\int_\Omega \frac{|u(x)(\phi(x)-\phi(y)) + \phi(y)(u(x)-u(y))|^p}{|x-y|^{N+ps}} \chi_{\{(x,y)||x-y|<1\}} dxdy \\
\le& C\|\phi\|^p_{C_B^{0,\gamma}(\overline{\Omega})}\left(\int_\Omega\int_\Omega \frac{|u(x)|^p}{|x-y|^{N-p(\gamma-s)}}\chi_{\{(x,y)||x-y|<1\}} dydx + \left([u]_{\Omega}^{s,p}\right)^p\right) \\
\le& C\|\phi\|^p_{C_B^{0,\gamma}(\overline{\Omega})}\left(\int_\Omega |u(x)|^p \int_{B_1(0)^c} \frac{1}{|z|^{N-p(\gamma-s)}} dzdx + \left([u]_{\Omega}^{s,p}\right)^p\right) \\
\le& C\|\phi\|^p_{C_B^{0,\gamma}(\overline{\Omega})}\left(\|u\|^p_{L^p(\Omega)} + \left([u]_{\Omega}^{s,p}\right)^p\right).
\end{aligned}$$

On the other hand, when $p = \infty$ we can use the decomposition (3.30)

$$[u\phi]_{\Omega}^{s,\infty} = \max\{I_1(u\phi, s, \infty), I_2(u\phi, s, \infty)\}.$$

As in the previous case, since $u\phi \in L^\infty(\Omega)$ it follows that:

$$I_2 \le C\|u\phi\|_{L^\infty(\Omega)} \le C\|\phi\|_{C_B^{0,\gamma}(\overline{\Omega})}\|u\|_{L^\infty(\Omega)},$$

whereas for the first term, in an analogous way to the previous case, we have

$$\begin{aligned}
I_1(u\phi, s, \infty) =& \operatorname*{ess\,sup}_{x,y\in\Omega, 0<|x-y|<1} \frac{|u(x)(\phi(x)-\phi(y)) - \phi(y)(u(x)-u(y))|}{|x-y|^s} \\
\le& C\|\phi\|_{C_B^{0,\gamma}(\overline{\Omega})}\left(\operatorname*{ess\,sup}_{x,y\in\Omega, 0<|x-y|<1} \frac{|u(x)|}{|x-y|^{s-\gamma}} + [u]_{\Omega}^{s,\infty}\right) \\
\le& C\|\phi\|_{C_B^{0,\gamma}(\overline{\Omega})}\left(\|u\|_{L^\infty(\Omega)} + [u]_{\Omega}^{s,\infty}\right),
\end{aligned}$$

concluding the proof. □

Corollary 50. *Let Ω be an open subset of $\mathbb{R}^N$, let $0 < s < \gamma \le 1$, $k \in \mathbb{N}_0$ and let $1 \le p \le \infty$. Given $u \in W^{k+s,p}(\Omega)$ and $\phi \in C_B^{k,\gamma}(\overline{\Omega})$ then $u\phi \in W^{k+s,p}(\Omega)$.*

Proof. The proof follows directly from *Proposition* 49 and [[2],Proposition 9.4]. □

Definition 51. *Given a function $\zeta \in C_0^\infty(\mathbb{R}^N)$ such that:*

$$\zeta(x) = \begin{cases} 1, & |x| \le | \\ 0, & |x| \ge 2. \end{cases}$$

we define the sequence of ***cut-off*** *functions $\{\zeta_n\}_{n\in\mathbb{N}}$ where $\zeta_n(x) := \zeta(\frac{x}{n})$.*

Corollary 52. *Let $s \in (0,1)$, $p \in [1,\infty)$ and $u \in W^{s,p}(\mathbb{R}^N)$. Then given a sequence of cut-off functions $\{\zeta_n\}_{n\in\mathbb{N}}$ we have that $\zeta_n u \in W^{s,p}(\mathbb{R}^N)$ and that*

$$\|\zeta_n u - u\|_{W^{s,p}(\mathbb{R}^N)} \to 0,$$

as $n \to \infty$, for any $u \in W^{s,p}(\mathbb{R}^N)$.

Proof. Let $u \in W^{s,p}(\mathbb{R}^N)$. It is clear that $|\zeta_n u - u| \le |u|$ for all $n \in \mathbb{N}$, also notice that $\{u\zeta_n\}_{n\in\mathbb{N}}$ converges a.e. to u. Hence, the dominated convergence theorem guarantees the convegence in $L^p(\mathbb{R}^N)$.
On the other hand, the mean value theorem implies that

$$|\zeta_n(x) - \zeta_n(y)| \le \frac{1}{n}\|\nabla\zeta\|_{L^\infty(\mathbb{R}^N)}|x-y|, \qquad \forall x, y \in \mathbb{R}^N, \tag{3.32}$$

implying that $\zeta_n \in C^{0,1}_B(\mathbb{R}^N)$ for all $n \in \mathbb{N}$. Therefore *Proposition* 49 guarantees that $\zeta_n u \in W^{s,p}(\mathbb{R}^N)$.
Let us consider the decomposition proposed in (3.28)

$$\left([(\zeta_n - 1)u\phi]^{s,p}_{\Omega}\right)^p = I_1((\zeta_n-1)u, s, p) + I_2((\zeta_n-1)u, s, p).$$

Since $I_2((\zeta_n-1)u,s,p) \le C\|\zeta_n u - u\|_{L^p(\mathbb{R}^N)}$, the second term converges to 0 by the convergence in $L^p(\mathbb{R}^N)$ previously discussed. To estimate $I_1((\zeta_n-1)u,s,p)$ we follow the same idea of *Proposition* 49 using $1-\zeta_n$ as our Lipschitz function

$$\begin{aligned} I_1((\zeta_n-1)u,s,p) \le& \frac{\|\nabla\zeta\|^p_{L^\infty(\mathbb{R}^N)}}{n^p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x-y|^{N-p(1-s)}}\chi_{\{(x,y)||x-y|<1\}}dxdy \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(1-\zeta_n(y))(u(x)-u(y))|^p}{|x-y|^{N+ps}}dxdy. \end{aligned}$$

The convergence of the first term follows from the proof of *Proposition* 49 where we showed that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x-y|^{N-p(1-s)}}\chi_{\{(x,y)||x-y|<1\}}dxdy < \infty.$$

For the second term let us notice that

$$\frac{|(1-\zeta_n(y))(u(x)-u(y))|^p}{|x-y|^{N+ps}} \le \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}, \qquad \text{a. e. in } \mathbb{R}^N \times \mathbb{R}^N,$$

therefore using the dominated convergence theorem in $\mathbb{R}^N \times \mathbb{R}^N$ the result follows. □

Definition 53. *A sequence of* ***mollifiers*** $\{\rho_n\}_{n\in\mathbb{N}}$ *is any sequence of positive functions defined on $\mathbb{R}^N$ such that*

$$\rho_n \in C_0^\infty(\mathbb{R}^N), \qquad \text{supp } \rho_n \subset \overline{B(0,1/n)}, \qquad \int_{\mathbb{R}^N}\rho_n = 1.$$

Lemma 54. *Let $s \in (0,1)$ and $p \in [1,\infty]$ and $u \in W^{s,p}(\mathbb{R}^N)$.*

(i) *If $1 \le p \le \infty$ and $f \in L^1(\mathbb{R}^N)$ we have that $f \star u \in W^{s,p}(\mathbb{R}^N)$ and*

$$\|f \star u\|_{W^{s,p}(\mathbb{R}^N)} \le \|f\|_{L^1(\mathbb{R}^N)} \|u\|_{W^{s,p}(\mathbb{R}^N)}.$$

(ii) *If $1 \le p < \infty$ and $\{\rho_n\}_{n\in\mathbb{N}}$ is a sequence of molifiers then*

$$\|\rho_n \star u - u\|_{W^{s,p}(\mathbb{R}^N)} \to 0, \qquad \text{as } n \to \infty. \tag{3.33}$$

Proof. (i) By Young's inequality the following estimate holds

$$\|f \star u\|_{L^p(\mathbb{R}^N)} \le \|f\|_{L^1(\mathbb{R}^N)} \|u\|_{L^p(\mathbb{R}^N)}.$$

In the case when p is finite we have that

$$\begin{aligned}\left([f \star u]^{s,p}_{\mathbb{R}^N}\right)^p &= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \int_{\mathbb{R}^N} |f \star u(z+y) - f \star u(y)|^p \, dy dz \\ &= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \|f \star (\tau_z u - u)\|^p_{L^p(\mathbb{R}^N)} dz \\ &\le \|f\|^p_{L^1(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \int_{\mathbb{R}^N} |u(y+z) - u(y)|^p \, dy dz \\ &\le \|f\|^p_{L^1(\mathbb{R}^N)} \left([u]^{s,p}_{\mathbb{R}^N}\right)^p,\end{aligned}$$

where we used the change of variables $z = x - y$ and Young's inequality.
For the case $p = \infty$ it follows that

$$\begin{aligned}[f \star u]^{s,\infty}_{\mathbb{R}^N} &= \operatorname*{ess\,sup}_{\substack{x,y\in\mathbb{R}^N \\ x\ne y}} \frac{|f \star u(x) - f \star u(y)|}{|x-y|^s} \\ &\le \operatorname*{ess\,sup}_{\substack{x,y\in\mathbb{R}^N \\ x\ne y}} \int_{\mathbb{R}^N} |f(z)| \frac{|u(x-z) - u(y-z)|}{|(x-z)-(y-z)|^s} dz \\ &\le \|f\|_{L^1(\mathbb{R}^N)} [u]^{s,\infty}_{\mathbb{R}^N}.\end{aligned}$$

(ii) If $p < \infty$ [[2], Theorem 4.22] guarantees that $\{\rho_n \star u\}_{n\in\mathbb{N}}$ converges to u in $L^p(\mathbb{R}^N)$. By (i) we have that for any $n \in \mathbb{N}$, $\rho_n \star u \in W^{s,p}(\mathbb{R}^N)$. Thus, using repeatedly the substitution $z = x - y$ and Minkowski's integral inequality we obtain

$$
\begin{aligned}
\left([\rho_n \star u - u]^{s,p}_{\mathbb{R}^N})\right)^p &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\rho_n \star u(x) - u(x) - \rho_n \star u(y) + u(y)\right|^p}{|x-y|^{N+sp}} dydx \\
&= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \int_{\mathbb{R}^N} \left|\rho_n \star u(z+y) - u(z+y) - \rho_n \star u(y) + u(y)\right|^p dydz \\
&= \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \rho_n(w)\big[(\tau_z u(y-w) - u(y-w)) - (\tau_z u(y) - u(y))\big] dw \right|^p dydz \\
&\leq \int_{\mathbb{R}^N} \frac{1}{|z|^{N+sp}} \left[\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left|\rho_n(w)[(\tau_z u(y-w) - u(y-w)) - (\tau_z u(y) - u(y))]\right|^p dy \right)^{\frac{1}{p}} dw \right]^p dz \\
&= \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \rho_n(w) \left(\int_{\mathbb{R}^N} \frac{|(\tau_z u(y-w) - u(y-w)) - (\tau_z u(y) - u(y))|^p}{|z|^{N+sp}} dy \right)^{\frac{1}{p}} dw \right]^p dz \\
&\leq \left[\int_{\mathbb{R}^N} \rho_n(w) \left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\tau_z u(y-w) - u(y-w)) - (\tau_z u(y) - u(y))|^p}{|z|^{N+sp}} dydz \right]^{\frac{1}{p}} dw \right]^p \\
&= \left[\int_{\mathbb{R}^N} \rho_n(w) \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left| \frac{u(x-w) - u(y-w)}{|(x-w)-(y-w)|^{\frac{N}{p}+s}} - \frac{u(x) - u(y)}{|x-y|^{\frac{N}{p}+s}} \right|^p dydx \right)^{\frac{1}{p}} dw \right]^p .
\end{aligned}
$$

If we define $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$
\varphi(x,y) := \frac{u(x) - u(y)}{|x-y|^{\frac{N}{p}+s}}, \qquad \text{for almost all } x, y \in \mathbb{R}^N,
$$

our chain of inequalities implies

$$
[\rho_n \star u - u]^{s,p}_{\mathbb{R}^N} \leq \int_{B_{\frac{1}{n}}(0)} \rho_n(w) \|\tau_{(w,w)}\varphi - \varphi\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} dw,
$$

where $\tau(w,w)$ is the shift operator in $\mathbb{R}^{2N}$. Since $u \in W^{s,p}(\mathbb{R}^N)$, $\varphi \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$. Using the continuity of the L^p norm with respect to the translations (see [[20], pag 245]), given $\varepsilon > 0$ we can find $\delta > 0$ such that for any $|w| < \delta$

$$
\|\tau_{(w,w)}\varphi - \varphi\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} < \varepsilon.
$$

Taking $N \in \mathbb{N}$ such that $\frac{1}{N} \leq \delta$ we have that for $n \geq N$ yields

$$
[\rho_n \star u - u]^{s,p}_{\mathbb{R}^N} \leq \int_{B_{\frac{1}{n}}(0)} \rho_n(w)\varepsilon = \varepsilon,
$$

and this completes the proof.

□

Furnished with these last results we can prove the following density result.

Theorem 55. *For any $s \in (0,1)$ and $p \in [1,\infty)$ the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$.*

Proof. Let $u \in W^{s,p}(\mathbb{R}^N)$, by the second part of *Lemma* 54 and by *Corollary* 52 it follows that $\zeta_n(\rho_n \star u) \in W^{s,p}(\mathbb{R}^N)$ where ζ_n and ρ_n represents the cut-off functions and the molllifiers (respectively) for any $n \in \mathbb{N}$. Let us notice $\zeta_n \in C_B^{0,1}(\mathbb{R}^N)$ and that there exist $M > 0$ such that for

$$\|\zeta_n\|_{C_B^{0,1}(\mathbb{R}^N)} \le M,$$

for every $n \in \mathbb{N}$. Using Proposition 49 it follows that

$$\begin{aligned}\|\zeta_n(\rho_n \star u) - u\|_{W^{s,p}(\mathbb{R}^N)} &\le \|\zeta_n[(\rho_n \star u) - u]\|_{W^{s,p}(\mathbb{R}^N)} + \|\zeta_n u - u\|_{W^{s,p}(\mathbb{R}^N)} \\ &\le M\|(\rho_n \star u) - u\|_{W^{s,p}(\mathbb{R}^N)} + \|\zeta_n u - u\|_{W^{s,p}(\mathbb{R}^N)}.\end{aligned}$$

Finally, the convergence to zero of the last expression of this chain of inequalities is guaranteed by *Lemma* 54 (for the first term) and by *Corollary* 52 (for the second term). □

Proposition 56. *Let Ω be an open subset of $\mathbb{R}^N$. Let $u \in W^{s,p}(\Omega)$ with $s \in (0,1)$, $1 \le p \le \infty$ and let $G : \mathbb{R} \to \mathbb{R}$ satisfying $G(0) = 0$ and*

$$|G(x) - G(y)| \le C|x - y|,$$

for all $x, y \in \mathbb{R}$ and for some $C > 0$. Implying that $G(u) \in W^{s,p}(\Omega)$.

Proof. Since $|G(u)| \le C|u|$ it follows that $G(u) \in L^p(\Omega)$ for any $u \in L^p(\Omega)$.
The rest of the proof follows directly from the estimate

$$|G(u)(x) - G(u)(y)| \le C|u(x) - u(y)|, \qquad \text{a. e. } x, y \in \Omega.$$

implying that

$$[G(u)]_\Omega^{s,p} \le C[u]_\Omega^{s,p}.$$

□

The following proposition shows that the fractional differentiability is preserved under Lipschitz change of variables. This proof follows the ideas of [11].

Proposition 57. *Let Ω be an open subset of $\mathbb{R}^N$, $s \in (0,1)$ and $1 \le p \le \infty$. Let $\Psi : \Omega \to \mathbb{R}^N$ be a Lipschitz bijective function whose inverse is Lipschitz too. Then, for $u \in W^{s,p}(\Omega)$ we have that $u \circ \Psi^{-1} \in W^{s,p}(\Psi(\Omega))$. Moreover, in this case there exist a constant $C > 0$ (independent of u) such that $\|u \circ \Psi^{-1}\|_{W^{s,p}(\Psi(\Omega))} \le C\|u\|_{W^{s,p}(\Omega)}$.*

Proof. Let us consider first the case where p is finite. By *Theorem* 14 it follows that $u \circ \Psi^{-1}$ is measurable and that $u \circ \Psi^{-1} 1 \in L^p(\Psi(\Omega))$.
Since $H(x) := |\det(Jac\Psi(x))| \in L^\infty(\Omega)$ we can apply again *Theorem* 14 with the bi-Lipschitz function $(x, y) \to (\Psi(x), \Psi(y))$ to get:

$$\begin{aligned}\int_{\Psi(\Omega)}\int_{\Psi(\Omega)} \frac{|u(\Psi^{-1}(x)) - u(\Psi^{-1}(y))|^p}{|x - y|^{N+ps}} dxdy &= \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^p}{|\Psi(x) - \Psi(y)|^{N+ps}} H(x)H(y)dxdy \\ &\le C\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy,\end{aligned}$$

where in the last inequality we used the Lipschitz condition of the inverse of Ψ and the boundedness of H.

For the case $p=\infty$, it is clear that $u\chi_{K_n} \in L^1(\Omega)$ for $K_n := \Psi(\Omega)\cap B_n(0,1)$ with $n\in\mathbb{N}$. Therefore, by *Theorem* 14 it follows that $u\chi_{K_n}\circ\Psi^{-1}$ is measurable for any $n\in\mathbb{N}$ implying that $u\circ\Psi^{-1}$ is measurable on Ω. On the other hand, it is clear that $\|u\circ\Psi^{-1}\|_{L^\infty(\Psi(\Omega))} = \|u\|_{L^\infty(\Omega)}$.

The rest of the proof follows from using the Lipschitz condition of Ψ^{-1} and by noticing that:

$$[u\circ\Psi^{-1}]^{s,\infty}_{\Psi(\Omega)} = \operatorname*{ess\,sup}_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|\Psi(x)-\Psi(y)|^s} \le \operatorname*{ess\,sup}_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{C^s|x-y|^s} \le C[u]^{s,\infty}_{\Omega}.$$

□

3.3 Extension operators

Though motivated the *Theorem* 37, the relationship of the spaces $W^{k+s,2}(\Omega)$ with their version in the whole space, $H^{k+s}(\mathbb{R}^N)\cap L^2(\Omega)$, is a mere analogy. However, imposing certain regularity assumptions in the domain Ω it can be shown that $W^{k+s,2}(\Omega)$ is formed exclusively by the restriction to Ω of functions in $H^{k+s}(\mathbb{R}^N)\cap L^2(\Omega)$. In order to address this problem it is necessary to generalize *Theorem* 12 to the fractional case.
Due to the complexity of this extension problem we present its proof into steps which closely follows the approach given in the section 5 of [11]. We start extending functions supported inside Ω.

Lemma 58. *Let $1\le p<\infty$, let $s\in(0,1)$ and let Ω be an open subset of $\mathbb{R}^N$. Given $u\in W^{s,p}(\Omega)$ with supp$(u)\subset\subset\Omega$ then the zero extension:*

$$\hat{u}(x) := \begin{cases} u(x), & \text{if} \quad x\in\Omega,\\ 0, & \text{if} \quad x\notin\Omega,\end{cases}$$

belongs to $W^{s,p}(\mathbb{R}^N)$ and $\|\hat{u}\|_{W^{s,p}(\mathbb{R}^N)} \le C\|u\|_{W^{s,p}(\Omega)}$.

Proof. Given that $K :=$supp$(u)\subset\subset\Omega$, then $d := d(K,\Omega^c)>0$ implying

$$\begin{aligned}\left([\hat{u}]^{s,p}_{\mathbb{R}^N}\right)^p &= \int_\Omega\int_\Omega \frac{|\hat{u}(x)-\hat{u}(y)|^p}{|x-y|^{N+ps}}dxdy + 2\int_\Omega\int_{\Omega^c} \frac{|\hat{u}(x)-\hat{u}(y)|^p}{|x-y|^{N+ps}}dxdy\\ &\le \left([u]^{s,p}_\Omega\right)^p + 2\int_\Omega |u(y)|^p \int_{|x-y|\ge d} \frac{1}{|x-y|^{N+ps}}dxdy\\ &= \left([u]^{s,p}_\Omega\right)^p + C\|u\|^p_{L^p(\Omega)}.\end{aligned}$$

Finally, since $\|\hat{u}\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\Omega)}$ the result follows. □

As in the classical extension problem in $W^{n,p}$, the next step is to select a local model for well behaved domains, so the following lemma address the extension problem for functions defined in the upper half space or, in general, for symmetric domains with respect to the last component.

Lemma 59. *Let Ω be an open subset of $\mathbb{R}^N$ symmetric with respect to the last component and let us define the sets $\Omega_+ := \{(x', x_N) \in \Omega | x_N > 0\}$ and $\Omega_- := \{(x', x_N) \in \Omega | x_N \leq 0\}$. Then, given $u \in W^{s,p}(\Omega_+)$ with $s \in (0,1)$ and $1 \leq p < \infty$, then the symmetric extension:*

$$\hat{u}(x) := \begin{cases} u(x', x_N), & \text{if} \quad x_N > 0, \\ u(x', -x_N), & \text{if} \quad x_N < 0, \end{cases}$$

belongs to $W^{s,p}(\Omega)$ and $\|\hat{u}\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega_+)}$.

Proof. Analogously to the last lemma we have:

$$\begin{aligned}
\left([\hat{u}]_{\Omega}^{s,p}\right)^p =& 2\int_{\Omega_+}\int_{\Omega_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + 2\int_{\Omega_+}\int_{\Omega_-} \frac{|\hat{u}(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \\
\leq& 2\int_{\Omega_+}\int_{\Omega_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + 2\int_{\Omega_+}\int_{\Omega_+} \frac{|\hat{u}(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \\
=& 4\left([u]_{\Omega_+}^{s,p}\right)^p,
\end{aligned}$$

where the inequality comes from the fact that $(x_N + y_N)^2 \geq (x_N - y_N)^2$ for $x_N, y_N \geq 0$. On the other hand, since $\|\hat{u}\|_{L^p(\Omega)} = 2\|u\|_{L^p(\Omega_+)}$, taking the constant $C = 4$ the result follows. □

Remark 60. *In Lemma 58 and in Lemma 59 the extension processes were made in a linear way. This means, given $\alpha, \beta \in \mathbb{R}$ and u and v functions such that both of them fulfills the hypotheses of either Lemma 58 or Lemma 59 then the extension satisfies $\widehat{\alpha u + \beta v} = \alpha \hat{u} + \beta + \hat{v}$.*

Furnished with these two lemmas we can present the general extension result.

Theorem 61. *Let $p \in [1, \infty)$, let $s \in (0,1)$ and let Ω be an open set of class $C^{0,1}$ in $\mathbb{R}^N$ such that $\partial\Omega$ is bounded . Then there exist a linear extension operator, i. e.*

$$\begin{aligned} T : W^{s,p}(\Omega) &\to W^{s,p}(\mathbb{R}^N) \\ u &\to Tu, \end{aligned}$$

with $Tu(x) = u(x)$ a. e. $x \in \Omega$ such that

$$\|Tu\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{L^p(\Omega)}, \tag{3.34}$$

and

$$\|Tu\|_{W^{s,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{s,p}(\Omega)}. \tag{3.35}$$

Proof. Let $u \in W^{s,p}(\Omega)$. Since Ω is an open set of class $C^{0,1}$ and $\partial\Omega$ is compact there exist a finite collection of open balls $\{B_{r_i}(x_i)\}_{i=1}^n$ with $r_i > 0$ and $x_i \in \partial\Omega$ covering $\partial\Omega$, and there exist functions $\Psi_i : Q \to B_{r_i}(x_i)$ for $i = 1, \cdots, n$ as in the *Definition* 10.
Let us define $U_0 := \Omega$ and $U_i := B_{r_i}(x_i)$ for $i = 1, \cdots, n$; in virtue of *Theorem* 11 there exist a partition of unity $\{\phi_i\}_{i=0}^n$ of class $C_0^\infty(\mathbb{R}^N)$ subordinated to $\{U_i\}_{i=0}^n$.
First of all, let us consider the function $\phi_0 u$. Since the derivative of ϕ_0 is bounded it follows that $\phi_0 \in C_B^{0,1}(\overline{\Omega})$. Therefore, combining the *Proposition* 56 and the *Lemma* 58 we have

that $\phi_0 u \in W^{s,p}(\Omega)$ and is compactly supported in Ω implying that it can be extended to a function $v_0 \in W^{s,p}(\mathbb{R}^N)$ such that

$$\|v_0\|_{W^{s,p}(\mathbb{R}^N)} \le C\|\phi_0 u\|_{W^{s,p}(\Omega)} \le \|\phi_0\|_{C_B^{0,1}(\overline{\Omega})}\|u\|_{W^{s,p}(\Omega)}. \tag{3.36}$$

Now, let us consider the function $u\circ\Psi_i : Q_+ \to \mathbb{R}$ for $i = 1, \cdots, n$. Clearly $u \in W^{s,p}\left(B_{r_i}(x_i)\cap\Omega\right)$ for $i = 1, \cdots, n$, therefore *Proposition* 57 guarantees that $u\circ\Psi_i \in W^{s,p}(Q_+)$ with

$$\|u\circ\Psi_i\|_{W^{s,p}(Q_+)} \le C\|u\|_{W^{s,p}(\Omega)}, \tag{3.37}$$

for $i = 1, \cdots, n$.
Since Q is symmetric with respect to the last component, *Lemma* 59 provide us an extension $w_i \in W^{s,p}(Q)$. Applying again *Proposition* 57 if follows that $w_i \circ \Psi_i^{-1} \in W^{s,p}(B_{r_i}(x_i))$ and combinig this with (3.37) it follows that

$$\|w_i\circ\Psi_i^{-1}\|_{W^{s,p}(B_{r_i}(x_i))} \le C\|u\|_{W^{s,p}(\Omega)}. \tag{3.38}$$

Applying the same argument used to extend $u\phi_0$ it follows that $\phi_i w_i \circ \Psi_i^{-1}$ can be extended to a function $v_i \in W^{s,p}(\mathbb{R}^N)$ such that:

$$\|v_i\|_{W^{s,p}(\mathbb{R}^N)} \le C\|\phi_i\|_{C_B^{0,1}(\overline{\Omega})}\|w_i\circ\Psi_i^{-1}\|_{W^{s,p}(B_{r_i}(x_i))} \le C\|u\|_{W^{s,p}(\Omega)}. \tag{3.39}$$

We claim that $Tu := \sum_{i=0}^n v_i$ give us the desired extension. Indeed, (3.36) and (3.39) implies (3.35) and since all the inqualities proved here are also valid changing the $W^{s,p}$ norm by the respective L^p norm (3.34) also holds.
In order to check that is an extension notice that for almost all $x \in \Omega$ it holds that:

$$\sum_{i=0}^n v_i(x) = \phi_0(x)u(x) + \sum_{i=1}^n \phi_i(x) w_i(\Psi_i^{-1}(x)) = \phi_0(x)u(x) + \sum_{i=1}^n \phi_i(x) u(\Psi\circ\Psi_i^{-1}(x)) = u(x).$$

The linearity follows from *Remark* 60 and from noticing that all the steps used to extend u were linear in the same sense explained in this remark.

□

Corollary 62. *Let $p \in [1,\infty)$, let $s \in (0,1)$ and let Ω be a open set of class $C^{0,1}$ in $\mathbb{R}^N$ such that $\partial\Omega$ is bounded. Then given $u \in W^{s,p}(\Omega)$ there exist a sequence of functions $\{\phi\}_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^N)$ such that the sequence of restrictions $\{\phi|_\Omega\}_{n\in\mathbb{N}}$ converges to u in $W^{s,p}(\Omega)$.*

Proof. Using *Theorem* 61 we can extend u to a function $Tu \in W^{s,p}(\mathbb{R}^N)$. On the other hand, *Theorem* 55 implies that there exist a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^N)$ that converges to Tu in $W^{s,p}(\mathbb{R}^N)$. Hence, $\{\phi_n|_\Omega\}_{n\in\mathbb{N}}$ is our desired sequence. □

Now we are in conditions to draw the direct realtionship between the spaces $W^{s,p}(\Omega)$ and $H^s(\mathbb{R}^N)$. From this last part of the section we follow Chapter 7 of [1].

Definition 63. *Let Ω be an open subset of $\mathbb{R}^N$ and let $s > 0$ then let us define*

$$H^s(\Omega) = \left\{u|_\Omega \quad \middle| u \in H^s(\mathbb{R}^N)\right\} / \sim,$$

where $u \sim v$ if and only if $u(x) = v(x)$ a.e. $x \in \Omega$. The equivalence class of u is represented by $[u]$.
This space is endowed with the norm

$$\|[u]\|_{H^s(\Omega)} = \inf_{v \sim u} \|v\|_{H^s(\mathbb{R}^N)}. \tag{3.40}$$

Remark 64. *Let us consider the set*

$$Z := \left\{u \in H^s(\mathbb{R}^N) \middle| \quad u = 0 \quad a.\ e. \quad in\ \Omega\right\},$$

clearly Z is a closed subspace of $H^s(\mathbb{R}^N)$. By construction, it follows that $H^s(\Omega)$ is equal to the quotient $H^s(\mathbb{R}^N)/Z$ which is a Banach space in virtue of the closedness of Z and it is endowed with the quotient norm. In this case the quotient norm is given by

$$\|u\|_{H^s(\mathbb{R}^N)/Z} := \inf_{v \in Z} \|u - v\|_{H^s(\mathbb{R}^N)} = \|P_{Z^\perp} u\|_{H^s(\mathbb{R}^N)},$$

where $P_{Z^\perp} u$ represents the projection of u onto the orthogonal complement of Z in $H^s(\mathbb{R}^N)$. By construction it is clear that this quotient norm coincides with $\|\cdot\|_{H^s(\Omega)}$. Hence, the norms in $H^s(\Omega)$ can be computed via an inner product using the following formula

$$\|u\|^2_{H^s(\Omega)} = \langle P_{Z^\perp} u, P_{Z^\perp} u\rangle_{H^s(\mathbb{R}^N)}.$$

These considerations shows that $H^s(\Omega)$ is a Hilbert space.

The next proposition proves a claim in the Chapter 7 of [1] which shows that $W^{s,2}(\Omega)$ is essentially the subspace of real-valued functions in $H^s(\Omega)$.

Proposition 65. *Let $s \in (0,1)$ and let Ω be an open subset of $\mathbb{R}^N$ with bounded boundary. If Ω is an open set of class $C^{0,1}$, then for any $[u] \in H^s(\Omega)$ with u real-valued we have that $u|_\Omega \in W^{s,2}(\Omega)$. Conversely, for any $v \in W^{s,2}(\Omega)$ there exist an extension $Tv \in H^s(\mathbb{R}^N)$. Moreover, keeping this notation there exist constants $C, C' > 0$ such that:*

$$\|u\|_{H^s(\Omega)} \leq C \|u|_\Omega\|_{W^{s,2}(\Omega)},$$

and

$$\|v\|_{W^{s,2}(\Omega)} \leq C' \|[Tv]\|_{H^s(\Omega)}.$$

Proof. Given $v \in W^{s,2}(\Omega)$, in virtue of the proof of *Theorem* 61 there exist a real-valued extension $Tv \in H^s(\mathbb{R}^N)$ (by *Theorem* 37), therefore $v = Tv|_\Omega \in H^s(\Omega)$ and

$$\|[Tv]\|_{H^s(\Omega)} \leq \|Tv\|_{H^s(\mathbb{R}^N)} \leq C\|v\|_{W^{s,2}(\Omega)}.$$

Conversely, given $[u] \in H^s(\Omega)$, u real-valued, *Theorem* 37 guarantees that

$$\|u\|_{W^{s,2}(\Omega)} \leq \|v\|_{H^s(\mathbb{R}^N)},$$

for any $v \in H^s(\mathbb{R}^N)$ such that $[u] = [v]$, hence it follows that $\|u\|_{W^{s,2}(\Omega)} \leq \|u\|_{H^s(\Omega)}$. □

3.4 Sobolev embeddings and some consequences

In this section we present an analogous result of the Sobolev embeddings for the fractional Sobolev spaces.

The following theorem is called **fractional Sobolev embedding** and it is the fractional analogous of the well-known Sobolev-Gagliardo-Niremberg theorem, see [[2], *Theorem* 9.9]. We present here a simple (and elegant) proof of this theorem due to H. Brezis, also see [[27], *Proposition* 15.5].

Theorem 66. *Let $s \in (0,1)$. If $1 \le p < \frac{N}{s}$, then the inclusion $W^{s,p}(\mathbb{R}^N) \subset L^{\frac{Np}{N-sp}}(\mathbb{R}^N)$ is continuous. Moreover, for $u \in W^{s,p}(\mathbb{R}^N)$, we have*

$$\|u\|_{L^{\frac{Np}{N-sp}}(\mathbb{R}^N)} \le C[u]^{s,p}_{\mathbb{R}^N}, \tag{3.41}$$

for some positive constant $C > 0$ depending only on N, s and p.

Proof. Let us assume first that $u \in W^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For a. e. $x \in \mathbb{R}^N$ and any $r > 0$ we have that:

$$|u(x)| \le \frac{1}{|B_r(x)|}\int_{B_r(x)} |u(x)-u(y)|dy + \frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y)|dy,$$

where $|B_r(x)|$ denotes the Lebesgue's measure in $\mathbb{R}^N$ of the ball $B_r(x)$. Applying Hölder inequality in the first term with p and in the second term with $q := \frac{Np}{N-sp}$ for both terms of the right hand we get

$$|u(x)| \le \frac{r^{-\frac{N}{p}}}{|B_1(x)|^{\frac{1}{p}}}\left(\int_{B_r(x)} |u(x)-u(y)|^p dy\right)^{\frac{1}{p}} + \frac{r^{-\frac{N}{q}}}{|B_1(x)|^{\frac{1}{q}}}\left(\int_{B_r(x)} |u(y)|^q dy\right)^{\frac{1}{q}}.$$

Multiplying and dividing the first term of last expression by $|x-y|^{N+sp}$ it follows that

$$|u(x)| \le \frac{r^{s}}{|B_1(x)|^{\frac{1}{p}}}\left(\int_{B_r(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dy\right)^{\frac{1}{p}} + \frac{r^{-\frac{N}{q}}}{|B_1(x)|^{\frac{1}{q}}}\left(\int_{B_r(x)} |u(y)|^q dy\right)^{\frac{1}{q}}.$$

Hence, this inequality implies, in particular, that

$$|u(x)| \le \frac{r^{s}}{|B_1(x)|^{\frac{1}{p}}}\left(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dy\right)^{\frac{1}{p}} + \frac{r^{-\frac{N}{q}}}{|B_1(x)|^{\frac{1}{q}}}\left(\int_{\mathbb{R}^N} |u(y)|^q dy\right)^{\frac{1}{q}}. \tag{3.42}$$

Since $u \in W^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $q \in (p,\infty)$ it follows that the right hand side of (3.42) is finite x a.e. in $\mathbb{R}^N$.
Minimizing the right hand side of (3.42) with respect to r we get

$$|u(x)| \le C\left(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dy\right)^{\frac{N}{Np+sqp}} \left(\int_{\mathbb{R}^N} |u(y)|^q dy\right)^{\frac{s}{N+sq}}, \tag{3.43}$$

with $C > 0$ depending only on s, N and p.
Using the definition of q we can rewrite (3.43) as

$$|u(x)| \leq C \left(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |u(y)|^q dy \right)^{\frac{s}{N+sq}}.$$

Raising both sides to the q and integrating yields

$$\int_{\mathbb{R}^N} |u(y)|^q dy \leq C \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dy dx \right) \left(\int_{\mathbb{R}^N} |u(y)|^q dy \right)^{\frac{qs}{N+sq}}.$$

Finally, we get (3.41) dividing both sides by $\left(\int_{\mathbb{R}^N} |u(y)|^q dy \right)^{\frac{qs}{N+sq}}$.
In the general case, for $u \in W^{1,p}(\mathbb{R}^N)$ we can use *Corollary* 52 to approximate it in $W^{1,p}(\mathbb{R}^N)$ by a sequence of functions given by $\zeta_n u \in W^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ where $\{\zeta_n\}_{n\in\mathbb{N}}$ is a sequence of cut-off functions. Combining this fact with Fatou's lemma the result follows.

□

Corollary 67. *Let $s \in (0,1)$ and let Ω be a Lipschitz bounded domain. If $1 \leq p < \frac{N}{s}$, then the inclusion $W^{s,p}(\Omega) \subset L^q(\Omega)$ is continuous for $q \in \left[1, \frac{Np}{N-sp}\right]$.*

Proof. The proof follows directly from applying the previous theorem and *Theorem* 61. □

The following result provides the corresponding version of the compact embeddings for the fractional Sobolev spaces.

Theorem 68. *Let $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp < N$. Let $q \in \left[1, \frac{Np}{N-sp}\right)$, Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $\mathcal{G}$ be a bounded set in $L^p(\Omega)$. Suppose that*

$$\sup_{u\in\mathcal{G}} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dx dy < \infty. \tag{3.44}$$

Then $\mathcal{G}$ is precompact in $L^q(\Omega)$.

Proof. See chapter 7 of [11]. □

4. Equivalent definitions and basic properties of the fractional Laplacian

In section 2.3 we pointed out how the fractional Laplacian operator $(-\Delta)^s$ characterizes the Sobolev spaces $H^s(\mathbb{R}^N)$ (see *Remark* 35), analyzing the differentiability properties of the functions by means of the Fourier transform. On the other hand, the techniques developed in the last chapter allowed us to translate the differentiablity conditions of the $H^s(\mathbb{R}^N)$ spaces into a more classical condition on the quotient of differences (or modulus of continuity). Hence, in the same spirit of *Theorem* 37 we present an equivalent definition for the fractional Laplacian that does not involve the Fourier transform and which also shows that the fractional Laplacian of functions in $\mathscr{S}(\mathbb{R}^N)$ is real-valued.

First, let us introduce an useful notation.

Definition 69. *Given $u : \mathbb{R}^N \to \mathbb{R}$ and $y \in \mathbb{R}^N$ we define the second differences operator δ as*

$$\delta u(x,y) = 2u(x) - u(x+y) - u(x-y).$$

The proof of the following result is based on the ideas presented in [[11], Proposition 3.3].

Theorem 70. *Let $s \in (0,1)$ and let $(-\Delta)^s$ be the fractional Laplacian operator defined by (2.13). Then, for $\varphi \in \mathscr{S}(\mathbb{R}^N)$,*

$$(-\Delta)^s \varphi(x) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \frac{\delta\varphi(x,y)}{|y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N, \tag{4.45}$$

where $C(N,s)$ is a constant that only depends on N and s.

Proof. By the injectivity of the Fourier transform in $L^1(\mathbb{R}^N)$ if suffices to show that

$$\mathscr{F}\Big((-\Delta)^s\varphi\Big) = \frac{C(N,s)}{2}\mathscr{F}\Big(\int_{\mathbb{R}^N} \frac{\delta\varphi(\cdot,y)}{|y|^{N+2s}} dy\Big). \tag{4.46}$$

On the other hand, let us consider:

$$\left|\frac{\delta\varphi(x,y)}{|y|^{N+2s}}\right| = \underbrace{\left|\frac{\delta\varphi(x,y)}{|y|^{N+2s}}\right|\chi_{B_1(0)^c}(y)}_{P_1} + \underbrace{\left|\frac{\delta\varphi(x,y)}{|y|^{N+2s}}\right|\chi_{B_2(0)\times B_1(0)}(x,y)}_{P_2} + \underbrace{\left|\frac{\delta\varphi(x,y)}{|y|^{N+2s}}\right|\chi_{B_2(0)^c\times B_1(0)}(x,y)}_{P_3}.$$

Clearly, Fubini-Tonelli's theorem guarantees that $P_1 \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. The integrability of P_2 follows from Taylor's formula with Lagrange remainder

$$P_2 \le \frac{\|D^2\varphi\|_{L^\infty(\mathbb{R}^N)}}{|y|^{N-2(1-s)}}\chi_{B_2(0)\times B_1(0)}(x,y).$$

Finally, for P_3, applying Taylor's formula with integral remainder yields

$$\begin{aligned} P_3 &\le \left|\frac{\int_0^1 |D^2\varphi(x+ty)| + |D^2\varphi(x+ty)| dt}{|y|^{N-2(1-s)}}\right|\chi_{B_2(0)^c\times B_1(0)}(x,y) \\ &\le 2\Big(\sup_{z\in\mathbb{R}^N} |D^2\varphi(z)||z|^M\Big)\frac{\chi_{B_2(0)^c\times B_1(0)}(x,y)}{|x|^M|y|^{N-2(1-s)}}, \end{aligned}$$

where $M > 0$ is a free parameter. Thus, taking $M > N$ it follows that $P_3 \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. Hence, Fubini-Tonellli's theorem implies

$$\mathscr{F}\left(\int_{\mathbb{R}^N} \frac{2\varphi(\cdot) - \varphi(\cdot + y) - \varphi(\cdot - y)}{|y|^{N+2s}} dy\right)(\xi) = \int_{\mathbb{R}^N} \frac{\mathscr{F}(\varphi)(\xi)(2 - e^{2\pi y\cdot\xi} - e^{-2\pi y\cdot\xi})}{|y|^{N+2s}} dy$$

$$= \mathscr{F}(u)(\xi) \int_{\mathbb{R}^N} \frac{4\sin^2(\pi\xi\cdot y)}{|y|^{N+2s}} dz = C\mathscr{F}(u)(\xi)|\xi|^{2s},$$

with

$$C = \int_{\mathbb{R}^N} \frac{4\sin^2(\pi y_1)}{|y|^{N+2s}} dy,$$

as in *Theorem* 37. Which finishes the proof. □

Remark 71. *Using the change of variable $z = 2\pi y$ we obtain*

$$C = 2(2\pi)^{2s} \int_{\mathbb{R}^N} \frac{2\sin^2(\frac{z_1}{2})}{|z|^{N+2s}} dz,$$

implying that

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{2\sin^2(\frac{z_1}{2})}{|z|^{N+2s}} dz\right)^{-1} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(z_1)}{|z|^{N+2s}} dz\right)^{-1}.$$

Remark 72. *Let us remember the classical Poisson equation*

$$-\Delta u(x) = f(x), \qquad x \in \Omega, \tag{4.47}$$

where Ω is an open subset of $\mathbb{R}^N$ and $f \in C(\Omega)$. A function $u : \Omega \to \mathbb{R}$ is called a classical solution of (4.47) *if $u \in C^2(\Omega)$ and satisfies* (72)*. Let us notice that this notion of solution does not require any integrability assumption neither in u nor in its derivates, unlike the strong solutions of the problem* (2.14) *or the weak solutions of the problem* (2.15)*. In the same spirit, we want to formulate a Poisson equation for the s-fractional Laplacian, namely*

$$(-\Delta)^s u(x) = f(x), \qquad x \in \Omega, \tag{4.48}$$

where $(-\Delta)^s$ is the formulation of the fractional Laplacian introduced in (4.45)*. We would also want to define a notion of solution of* (4.48) *that only involves conditions in the regularity of u, however the nonlocal structure of the equation imposes an extra condition on the integrability of u. We will discuss this issue extensively in the following section.*

4.1 Strong and distributional formulations of the fractional Laplacian

We start this section discussing the integrability condition, previously commented, that a function u has to satisfy in order to compute its fractional Laplacian understood as in (4.45).

Definition 73. *Let $s \in (0,1)$. We define the set*

$$L_s := \left\{ f \in \mathscr{M}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} \frac{|f(x)|}{1+|x|^{N+2s}} dx < \infty \right\}. \tag{4.49}$$

Equivalently, $f \in L_s$ if and only if $f \in L^1(\mathbb{R}^N, d\mu)$ with $d\mu(x) := \frac{1}{1+|x|^{N+2s}} dx$.

Remark 74. *Formally, the space L_s represents the space of functions in $L^1_{loc}(\mathbb{R}^N)$ whose Fourier transform (in a distributional sense) is "not too singular" at the origin.*

The following result is intended to provide a rigorous proof of a comment made in the second page of [3].

Theorem 75. *Let $s \in (0,1)$ and let $(-\Delta)^s$ be the fractional Laplacian operator. Then, for any function $f \in L_s$ the linear functional*

$$T_f : \mathscr{S}(\mathbb{R}^N) \to \mathbb{R},$$
$$\varphi \to \int_{\mathbb{R}^N} f(-\Delta)^s \varphi,$$

belongs to $\mathscr{S}'(\mathbb{R}^N)$.

Proof. Clearly if $f \in L_s$, then f induces a functional in $\mathscr{S}'(\mathbb{R}^N)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^N)$. Since for any $\varphi \in \mathscr{S}(\mathbb{R}^N)$ all the versions of the fractional Laplacian presented so far are equivalent, we can take $(-\Delta)^s$ given by (4.45). Using again the decomposition proposed in the proof of *Theorem* 70 we get a. e. $x \in \mathbb{R}^N$

$$|f(x)(-\Delta)^s \varphi(x)| = |f(x)|P_1 + |f(x)|P_2 + |f(x)|P_3.$$

By the arguments used in the proof of *Theorem* 70 we get that

$$|f(x)|P_2 \le |f(x)| \frac{\|D^2\varphi\|_{L^\infty(\mathbb{R}^N)}}{|y|^{N-2(1-s)}} \chi_{B_2(0)\times B_1(0)}(x,y) \in L^1(\mathbb{R}^N),$$

thus,

$$\|fP_2\|_{L^1(\mathbb{R}^N)} \le C\|D^2\varphi\|_{L^\infty(\mathbb{R}^N)},$$

with C only depending on f, N and s. On the other hand, taking $M \ge N+2s$ (particularly, $M = N+2$ works) and using the fact that $f \in L_s$ we get

$$|f(x)|P_3 \le |f(x)| 2\Big(\sup_{z\in\mathbb{R}^N} |D^2\varphi(z)||z|^{N+2}\Big) \frac{\chi_{B_2(0)^c\times B_1(0)}(x,y)}{|x|^{N+2}|y|^{N-2(1-s)}} \in L^1(\mathbb{R}^N),$$

implying that

$$\|fP_3\|_{L^1(\mathbb{R}^N)} \le C\Big(\sup_{z\in\mathbb{R}^N} |D^2\varphi(z)||z|^{N+2}\Big),$$

and again, with C being a constant independent of φ.
For the term associated to P_1, we can use the decaying properties of φ and some linear changes of variables (i.e. $z = x - y$ and $z = x + y$) to get,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{B_1(0)^c} |f(x)|P_1(x,y)dydx \le C\Big(\sup_{z\in\mathbb{R}^N}|\varphi(z)||z|^{N+2}\Big)\int_{\mathbb{R}^N}\frac{|f(x)|}{1+|x|^{N+2s}}dx \\ + \int_{\mathbb{R}^N}\int_{B_1(0)^c}\frac{|f(z-y)|}{1+|z-y|^{N+2s}}\frac{(1+|z-y|^{N+2s})||\varphi(z)|}{|y|^{N+2s}}dydz \\ + \int_{\mathbb{R}^N}\int_{B_1(0)^c}\frac{|f(z+y)|}{1+|z+y|^{N+2s}}\frac{(1+|z+y|^{N+2s})||\varphi(z)|}{|y|^{N+2s}}dydz.\end{aligned}$$

Combining the latter inequality with the estimates for fP_2 and fP_3 we conclude that there exists $C > 0$ independent of φ such that

$$|T_f(\varphi)| \le C\bigg(\sup_{z\in\mathbb{R}^N}|D^2\varphi(z)|\Big(1+|z|^{N+2}\Big) + \sup_{z\in\mathbb{R}^N}|\varphi(z)|\Big(1+|z|^{N+2}\Big)\bigg), \tag{4.50}$$

implying that $T_f \in \mathscr{S}'(\mathbb{R}^N)$. □

The previous theorem allow us to give a distributional definition for the fractional Laplacian.

Definition 76. *Let $s \in (0,1)$. For $f \in L_s$ we define the distributional fractional Laplacian of f by $(-\Delta)^s f := T_f$, with T_f defined as in Theorem 75.*

Remark 77. *Usually, the distributional definitions allow us to extend differential operators to some functions which are not locally well-behaved. For instance, in (4.50) we can see that the first term involves the regularity of the second derivative of the test function, supplying in some sense the lack of local regularity of f. However, the distributional extension of this operator also requires a integrability condition imposed in the definition of L_s that can not be relaxed. This condition is necessary in the estimation of the term associated to P_1 in the Theorem 75. To illustrate this situation let us fix $f(x) = |x|^{N+2s}$ and $\varphi \in \mathscr{S}(\mathbb{R}^N)$ with $\varphi \ge 1$ in $B_1(0)$, therefore*

$$\int_{\mathbb{R}^N}\int_{B_1(0)^c}\frac{|f(z-y)||\varphi(z)|}{|y|^{N+2s}}dydz \ge \int_{B_1(0)}\int_{B_1(0)^c}\frac{|z-y|^{N+2s}}{|y|^{N+2s}}dydz = \infty. \tag{4.51}$$

Now we want to address the problem of establishing a set of local conditions that a function should satisfy in order to define its fractional Laplacian in a strong sense. A result of this type is proved in [37]. We present a different proof of this result using a more theoretic measure approach, which will require the following lemma.

Lemma 78. *Let μ be the finite Borel measure defined by*

$$\mu(A) := \int_A \frac{1}{1+|y|^{N+2s}}dy.$$

Then for every $f \in L^1(\mathbb{R}^N, d\mu)$ we have that

$$\lim_{h\to 0}\int_{\mathbb{R}^N}|\tau_h f - f|\,d\mu = 0.$$

Proof. Since μ is a finite Borel measure then μ is regular and $C_0(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N, d\mu)$ with its norm. Hence, given $\varepsilon > 0$ there exists $\varphi \in C_0(\mathbb{R}^N)$ such that $\|\phi - f\|_{L^1(\mathbb{R}^N,d\mu)} < \varepsilon$. Since φ is uniformly continuous there exists $\delta \in (0,1)$ such that for $|h| < \delta$ it holds that $|\varphi(y+h) - \varphi(y)| < \frac{\varepsilon}{\mu(\mathbb{R}^N)}$ for every $y \in \mathbb{R}^N$. Thus, using the change of variables $z = y + h$ we have that

$$\begin{aligned}\int_{\mathbb{R}^N} |\tau_h f - f|\, d\mu &\leq \int_{\mathbb{R}^N} \left|\tau_h f - \tau_h \varphi\right| d\mu + \int_{\mathbb{R}^N} \left|\tau_h \varphi - \varphi\right| d\mu + \int_{\mathbb{R}^N} \left|\varphi - f\right| d\mu \\ &< 2\varepsilon + \int_{\mathbb{R}^N} \frac{\left|f(z) - \varphi(z)\right|}{1 + |z-h|^{N+2s}} dz.\end{aligned}$$

Given that $|h| < 1$ we have for $|z| > 2$

$$|z - h| \geq |z| - 1 = |z|\left(1 - \frac{1}{|z|}\right) \geq \frac{|z|}{2}.$$

Thus there exists $C > 0$ depending only on N and s such that

$$\frac{1}{1 + |z-h|^{N+2s}} \leq C \frac{1}{1 + |z|^{N+2s}}, \qquad \forall h, |h| \leq 1.$$

Finally, we get that

$$\begin{aligned}\int_{\mathbb{R}^N} |\tau_h f - f|\, d\mu &\leq 2 + \varepsilon + C \frac{\left|f(z) - \varphi(z)\right|}{1 + |z|^{N+2s}} dz \\ &\leq (2 + C)\varepsilon,\end{aligned}$$

for all $\varepsilon > 0$, completing the proof. □

Proposition 79. *Given $s \in (0,1)$ and Ω an open subset of $\mathbb{R}^N$ we have that:*

(i) *If $s < \frac{1}{2}$, then for every $\varepsilon \in (0, 1-2s]$ and for every $u \in L_s$ such that $u|_\Omega \in C^{0,2s+\varepsilon}(\Omega)$, we have that $(-\Delta)^s u(x)$, understood as in (4.45), is well defined for any $x \in \Omega$ and $(-\Delta)^s u|_\Omega \in C(\Omega)$.*

(ii) *If $s \geq \frac{1}{2}$, then for every $\varepsilon \in (0, 2-2s]$ and for every $u \in L_s$ such that $u|_\Omega \in C^{1,2s-1+\varepsilon}(\Omega)$, $(-\Delta)^s u(x)$ (understood as in (4.45)) is well defined for any $x \in \Omega$ and $(-\Delta)^s u|_\Omega \in C(\Omega)$.*

Proof. (i) Let $\varepsilon \in (0, 1-2s]$ and $u \in L_s$ such that $u|_\Omega \in C^{0,2s+\varepsilon}(\Omega)$. Given $x \in \Omega$ there exists $\eta > 0$ such that $B_\eta(x) \subset \Omega$, hence for $y \in \mathbb{R}^N$ we have that,

$$\frac{|\delta u(x,y)|}{|y|^{N+2s}} \leq \frac{|\delta u(x,y)|}{|y|^{N+2s}} \chi_{(B_\eta(0))^c}(y) + 2C \frac{1}{|y|^{N-\varepsilon}} \chi_{B_\eta(0)}(y), \tag{4.52}$$

where $C > 0$ is the Hölder continuity modulus of u. On the other hand, we have that

$$\begin{aligned}\frac{|\delta u(x,y)|}{|y|^{N+2s}} \chi_{(B_\eta(0))^c}(y) &\leq 2|u(x)| \frac{1}{|y|^{N+2s}} \chi_{(B_\eta(0))^c}(y) \\ &\quad + K_x \left(\frac{|u(x+y)|}{1 + |x+y|^{N+2s}} + \frac{|u(x-y)|}{1 + |x-y|^{N+2s}} \right) \chi_{(B_\eta(0))^c}(y),\end{aligned}$$

where K_x is a positive constant independent of y.
Hence, we conclude that

$$\frac{|\delta u(x,y)|}{|y|^{N+2s}} \in L^1(\mathbb{R}^N), \quad \forall x \in \Omega.$$

Now, to prove the continuity of $(-\Delta)^s u$ in x, let us notice that given $\epsilon > 0$ there exists $\rho \in (0, \frac{\eta}{2})$ such that

$$\int_{B_\rho(0)} 2C \frac{1}{|y|^{N-\varepsilon}} dy < \frac{\epsilon}{2}.$$

Also notice that from (4.52) we have that for $h \in B_{\frac{\eta}{2}(0)}$

$$\int_{\mathbb{R}^N} \frac{|\delta[\tau_h u - u](x,y)|}{|y|^{N+2s}} dy \le \int_{\mathbb{R}^N} \frac{|\delta[\tau_h u - u](x,y)|}{|y|^{N+2s}} \chi_{(B_\rho(0))^c}(y) dy + \frac{\epsilon}{2}. \tag{4.53}$$

Using the Hölder continuity of u and considering the finite measure in $\mathbb{R}^N$ given by $d\mu(y) := \frac{1}{1+|y|^{N+2s}} dy$ it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|\delta[\tau_h u - u](x,y)|}{|y|^{N+2s}} \chi_{(B_\rho(0))^c}(y) dy \le 2C|h|^{2s+\varepsilon} \int_{(B_\rho(0))^c} \frac{1}{|y|^{N+2s}} dy \\ + \left(\frac{1+\rho^{N+2s}}{\rho^{N+2s}}\right) \int_{\mathbb{R}^N} |u(x+h+y) - u(x+y)| d\mu(y) \\ + \left(\frac{1+\rho^{N+2s}}{\rho^{N+2s}}\right) \int_{\mathbb{R}^N} |u(x+h-y) - u(x-y)| d\mu(y).\end{aligned}$$

Lemma 78 implies that there exists $\delta > 0$ such that for every $|h| < \delta$

$$\int_{\mathbb{R}^N} \frac{|\delta[\tau_h u - u](x,y)|}{|y|^{N+2s}} \chi_{(B_\rho(0))^c}(y) dy < \frac{\epsilon}{2}.$$

Finally, the result follows from this latter inequality and from (4.53).

(ii) This proof is almost identical to the first part, the only change consists in replacing the Hölder continuity condition used in the first part of this proof by the following inequality

$$|\delta u(x,y)| \le \int_0^1 |\nabla u(x+ty) - u(x-ty)|\,|y| dt \le C|y|^{2s+\varepsilon}, \quad \forall x \in \Omega, \quad \forall y \in \mathbb{R}^N,$$

where $C > 0$ is a constant independent of x and y.

□

We can present now a version of the fractional Laplacian directly related to the Gagliardo seminorm $[u]^{s,2}_{\mathbb{R}^N}$. The results discussed in this last part fo the section are quite standard see for instance [11].

Theorem 80. *Let Ω be an open subset of $\mathbb{R}^N$, $s \in (0,1)$ and let $(-\Delta)^s$ be the fractional Laplacian operator defined by* (4.45)*. Then for $u \in L_s$ such that*

(i) $u|_\Omega \in C^{0,\gamma}(\Omega)$ *with* $\gamma \in (2s, 1]$ *if* $s < \frac{1}{2}$,

(ii) $u|_\Omega \in C^{1,\gamma}(\Omega)$ *with* $\gamma \in (2s-1, 1]$ *if* $s \geq \frac{1}{2}$,

we have that

$$(-\Delta)^s u(x) = C(N,s) \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \quad \forall x \in \Omega. \tag{4.54}$$

Proof. Let $\varepsilon > 0$, using the changes of variables $z = x - y$ and $z = y - x$ we get

$$\int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = \int_{|z|>\varepsilon} \frac{u(x)-u(x-z)}{|z|^{N+2s}} dz,$$

and

$$\int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = \int_{|z|>\varepsilon} \frac{u(x)-u(x+z)}{|z|^{N+2s}} dz,$$

respectively. Adding up these two expressions it follows that

$$C(N,s) \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = \frac{C(N,s)}{2} \int_{|z|>\varepsilon} \frac{2u(x)-u(x+z)-u(x-z)}{|z|^{N+2s}} dz.$$

Finally, by *Theorem* 70 and Lebesgue's dominated convergence theorem the result follows. □

Remark 81. *Usually, the limit in* (4.54) *is changed by the abbreviation p.v. that means "Principal Value". This limit is necessary in this alternative formulation for the fractional Laplacian since the right hand side of* (4.54) *should not be absolutely integrable. Formally, let us fix* $x \in \Omega$ *and* $y \in \mathbb{R}^N$*, then for* $s \geq \frac{1}{2}$ *we can use the mean value theorem to approximate the integrand of the right hand side of* (4.54) *by*

$$\frac{u(x)-u(y)}{|x-y|^{N+2s}} \simeq \frac{\nabla u(\zeta_{xy}) \cdot (x-y)}{|x-y|^{N+2s}},$$

where ζ_{xy} *is some point on the line that joins* x *and* y*. Roughly speaking, this representation shows that the function* $\frac{\nabla u(\zeta_{xy}) \cdot (x-y)}{|x-y|^{N+2s}}$ *is "odd" in the symmetric domain* $(B_\varepsilon(x))^C$ *providing some cancellations that ensure the convergence near the origin.*
However, for $s < \frac{1}{2}$ *the integral given in* (4.54) *is, in fact, not singular since*

$$\frac{|u(x)-u(y)|}{|x-y|^{N+2s}} \leq \frac{C}{|x-y|^{N-\varepsilon}} \in L^1(B_1(x)).$$

Moreover, using this idea we can show that the fractional Laplacian corresponds to an integro-differential operator as defined in *Section* 2.1.

Proposition 82. *Let* Ω *be an open subset of* $\mathbb{R}^N$*,* $s \in (\frac{1}{2}, 1)$*,* $\gamma \in (2s-1, 1]$ *and let* $(-\Delta)^s$ *be the fractional Laplacian operator defined by* (4.45)*. Then, for* $u \in L_s$ *such that* $u|_\Omega \in C^{1,\gamma}(\Omega)$ *and for* $x \in \Omega$*,*

$$(-\Delta)^s u(x) = C(N,s) \int_{\mathbb{R}^N} \frac{u(x)-u(y)-\nabla u(x) \cdot (y-x) \chi_{\{(x,y)||x-y|<1\}}}{|x-y|^{N+2s}} dy. \tag{4.55}$$

Proof. First of all, let us notice that given, $x \in \mathbb{R}^N$ and $\varepsilon > 0$

$$\int_{|x-y|>\varepsilon} \frac{\nabla u(x)\cdot(y-x)\chi_{\{(x,y)||x-y|<1\}}}{|x-y|^{N+2s}}dy = 0,$$

given that the integrand is odd with respect to the domain. Hence, by *Theorem* 80 it follows that

$$(-\Delta)^s u(x) = C(N,s)\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} \frac{u(x)-u(y)-\nabla u(x)\cdot(y-x)\chi_{\{(x,y)||x-y|<1\}}}{|x-y|^{N+2s}}dy.$$

Thereby, in order to conclude (4.55), it suffices to show that the integrand of the right-hand side belongs to $L^1(\mathbb{R}^N)$. Indeed, since Ω is open there exists $\delta > 0$ such that $B_\delta(x) \subset \Omega$, hence for $|x-y| \geq \delta$ the integrability is guaranteed since $u \in L_s$, whereas for $|x-y| < \delta$ we can apply the fundamental theorem of calculus to get

$$\left|\frac{\int_0^1(\nabla u(xt+(1-t)y)-\nabla u(x))\cdot(y-x)dt}{|x-y|^{N+2s}}\right|\chi_{B_1(x)}(y) \leq \frac{C}{|x-y|^{N-1-\gamma}}\chi_{B_\delta(x)}(y) \in L^1(\mathbb{R}^N).$$

Finally, applying Lebegue's dominated convergence theorem the result follows. □

4.2 Some properties of weak version of the fractional Laplacian

In the last part of chapter 2 we discussed the analogies between the strong and the weak variational problems associated to the Laplacian and to the fractional Laplacian. We also pointed out, that under certain conditions, (2.16) and (2.17) are equations in the real setting and that (2.17) has variational form. With the theory developed in the last two chapters we are in conditions to prove these two claims.

Lemma 83. *For $s \in (0,1)$ and $(-\Delta)^s$ as in Definition 33 we have that:*

(i) *If $u \in W^{2s,2}(\mathbb{R}^N)$ then $(-\Delta)^s u \in L^2(\mathbb{R}^N)$. In particular $(-\Delta)^s u$ is also real-valued.*

(ii) *For $u,v \in W^{2s,2}(\mathbb{R}^N)$ we have that*

$$\int_{\mathbb{R}^N}(-\Delta)^s uv = \int_{\mathbb{R}^N} u(-\Delta)^s v = (2\pi)^{2s}\int_{\mathbb{R}^N}|\xi|^{2s}\mathscr{F}(u)(\xi)\overline{\mathscr{F}(v)}(\xi)d\xi.$$

In particular, for $u,v \in W^{2s,2}(\mathbb{R}^N)$ we have that

$$\int_{\mathbb{R}^N}|\xi|^{2s}\mathscr{F}(u)(\xi)\overline{\mathscr{F}(v)}(\xi)d\xi \in \mathbb{R}.$$

Proof. (i) Given $u \in W^{2s,2}(\mathbb{R}^N)$, *Proposition* 42 guarantees that $(-\Delta)^s u \in L^2(\mathbb{R}^N;\mathbb{C})$. On the other hand, given $\varphi \in \mathscr{S}(\mathbb{R}^N)$, *Theorem* 70 implies that $(-\Delta)^s\varphi$ is real-valued. Thus, by density (*Theorem* 55) we can find a sequence of functions $\{\varphi_n\}$ in $\mathscr{S}(\mathbb{R}^N)$ converging to u in $W^{2s,2}(\mathbb{R}^N)$, therefore by *Proposition* 42 it follows that

$$\|(-\Delta)^s(\varphi_n - u)\|_{L^2(\mathbb{R}^N;\mathbb{C})} \to 0, \quad \text{as} \quad n\to\infty,$$

implying that (up to a subsequence) $(-\Delta)^s\varphi_n(x) \to (-\Delta)^s(u)(x)$ x a.e. in $\mathbb{R}^N$ as $n\to\infty$.

(ii) Let $u, v \in W^{2s,2}(\mathbb{R}^N)$. Using the fact that $\mathscr{F}(\overline{h}) = \overline{\mathscr{F}^{-1}(h)}$ for any h in $L^2(\mathbb{R}^N;\mathbb{C})$ and using the self-adjointness of $\mathscr{F}$ in $L^2(\mathbb{R}^N;\mathbb{C})$ we get

$$\int_{\mathbb{R}^N} (-\Delta)^s uv = (2\pi)^{2s} \int_{\mathbb{R}^N} |\xi|^{2s} \mathscr{F}(u)(\xi)\overline{\mathscr{F}(v)}(\xi) d\xi = (2\pi)^{2s} \overline{\int_{\mathbb{R}^N} |\xi|^{2s} \overline{\mathscr{F}(u)}(\xi)\mathscr{F}(v)(\xi) d\xi}$$
$$= \int_{\mathbb{R}^N} u(-\Delta)^s v.$$

□

5. Properties of the Fractional Laplacian as a linear operator

In this chapter we are insterested in studying the nonlocal version of the linear problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.56}$$

where, from now on, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and $f : \mathbb{R} \to \mathbb{R}$ is some suitable function. Namely,

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Omega^c, \end{cases} \tag{5.57}$$

for $s \in (0,1)$. Notice that in this case we are requiring every *solution* u to be 0 in the complement of Ω. This condition is known as the *nonlocal homogeneus Dirichlet condition* and Ω^c is known as the *nonlocal boundary of* Ω. This change in the boundary condition naturally arises from the fact that any *solution* u must be defined in the whole space $\mathbb{R}^N$ (see also (4.45)).

We are interested in proving the existence of *weak solutions*for the problem (5.57). In the following section we will discuss the functional setting to precisely define a notion of weak solution and the solution operator associated to (5.57).

5.1 Functional setting and linear problem

A natural functional space to find the weak solutions of this problem is given the by following non-local version of $H_0^1(\Omega)$

$$H_\Omega^s(\mathbb{R}^N) := \left\{ u \in W^{s,2}(\mathbb{R}^N) \middle| u = 0 \text{ a.e } \Omega^c \right\}.$$

Notice that we are using $W^{s,2}(\mathbb{R}^N)$ instead of $H^s(\mathbb{R}^N)$ because the former, given our definition, is a real Banach space, whereas the latter is a complex Banach space and this difference has a central relevance since many of techniques to be applied make sense only when the functionals are defined on an ordered field like $\mathbb{R}$. It is not difficult to see $H_\Omega^s(\mathbb{R}^N)$ is a closed subspace of $W^{s,2}(\mathbb{R}^N)$ and hence, it inherits its properties as a Hilbert space with the respective norm. Aditionally, $H_\Omega^s(\mathbb{R}^N)$ preserves some important density properties of the classical Sobolev spaces such as

- If Ω is Lipschitz then $H_\Omega^s(\mathbb{R}^N)$ is the closure of $C_0^\infty(\Omega)$ in $W^{s,2}(\mathbb{R}^N)$.
- If Ω is Lipschitz and $s \neq \frac{1}{2}$ then $H_\Omega^s(\mathbb{R}^N)$ is the completion of $C_0^\infty(\Omega)$ with the norm $W^{s,2}(\Omega)$.

For the proof of these results see chapter 3 of [23], in particular proposition 3.33. On the other hand, *Theorem* 68 implies that $H^s_\Omega(\mathbb{R}^N)$ is compactly embedded in $L^q(\Omega)$ for $q \in [1,2^*)$, where $2^* = \frac{2N}{N-2s}$.

Another important feature of this space is that the functions contained in it satisfy a nonlocal version of the Poincaré inequality.

Theorem 84. *[Poincaré inequality] If Ω is Lipschitz then there exist a constant $C>0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C[u]^{s,2}_{\mathbb{R}^N}, \qquad \forall u \in H^s_\Omega(\mathbb{R}^N). \tag{5.58}$$

Proof. By the density result that we pointed out above it suffices to prove the result for any $\varphi \in C^\infty_0(\Omega)$. By the fractional Sobolev inequality (*Theorem* 66) it follows that

$$\|\varphi\|_{L^{2^*}(\Omega)} \leq C[\varphi]^{s,2}_{\mathbb{R}^N}.$$

The result follows from the previous inequality and by noticing that $L^2(\Omega)$ is continuosly embedded in $L^{2^*}(\Omega)$ because Ω has finite measure. □

Remark 85. *Theorem 84 implies that, in $H^s_\Omega(\mathbb{R}^N)$, the Gagliardo seminorm is actually a norm equivalent to the H^s-norm. Hence, in the sequel, given $u,v \in H^s_\Omega(\mathbb{R}^N)$ we will use the following conventions*

$$\|u\|_{H^s_\Omega(\mathbb{R}^N)} := [u]^{s,2}_{\mathbb{R}^N}$$
$$\langle u,v\rangle_{H^s_\Omega(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy$$

This is, in some sense, the analogous expression to the $H^1_0(\Omega)$-norm in the nonlocal case.

In this case some properties of the functional space are easier to prove that in the context of the classical Sobolev spaces, like the fact that the negative and positive parts of a weakly differentiable function are also weakly differentiable. However, some subtleties arise with the introduction of this inner product, one of them is that the negative and positive parts of a function are not longer orthogonal. The following result addresses these considerations in the particular case of $H^s_\Omega(\mathbb{R}^N)$, nonetheless some of the arguments apply for the spaces $W^{s,p}(\Omega)$.

Proposition 86. *If $u \in H^s_\Omega(\mathbb{R}^N)$ then $|u|, u^+, u^- \in H^s_\Omega(\mathbb{R}^N)$ and*

(i) $\||u|\|_{H^s_\Omega(\mathbb{R}^N)} \leq \|u\|_{H^s_\Omega(\mathbb{R}^N)}$ *and the equality holds if and only if either $u^+=0$ or $u^-=0$ a.e in $\mathbb{R}^N$.*

(ii) $\langle u^+,u^-\rangle_{H^s_\Omega(\mathbb{R}^N)} \leq 0$ *and they are orthogonal if and only if one of them is 0 a.e in $\mathbb{R}^N$.*

Proof. Notice that by the reversed triangle's inequality it follows that $||u(x)|-|u(y)||^2 \leq |u(x)-u(y)|^2$ for every $x,y \in \mathbb{R}^N$ implying that $|u| \in H^s_\Omega(\mathbb{R}^N)$ also that $\||u|\|_{H^s_\Omega(\mathbb{R}^N)} \leq \|u\|_{H^s_\Omega(\mathbb{R}^N)}$.

Since $u^+ = \frac{1}{2}(u + |u|)$ and $u^- = \frac{1}{2}(|u| - u)$ it follows that $u^+, u^- \in H^s_\Omega(\mathbb{R}^N)$. The second part of (i) will be proved after proving (ii).

Let us prove (ii). By a direct computation it follows that

$$\langle u^+, u^- \rangle_{H^s_\Omega(\mathbb{R}^N)} = -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} dxdy \leq 0,$$

which is 0 only if either u^+ or u^- are 0 a.e. in $\mathbb{R}^N$. The rest of the proof of (i) follows from noticing that

$$\|u\|^2_{H^s_\Omega(\mathbb{R}^N)} - \||u|\|^2_{H^s_\Omega(\mathbb{R}^N)} = -4\langle u^+, u^- \rangle_{H^s_\Omega(\mathbb{R}^N)}.$$

□

Remark 87. *If u is a suitable continuous function in Ω such that $(-\Delta)^s u$ is defined a.e. in Ω (for instance, if u satisfies the hypotheses of Theorem 80) and u solves pointwise a.e. problem (5.57) for some $f \in L^2(\Omega)$, we can multiply both sides of the first equation by a test function $\varphi \in C_0^\infty(\Omega)$ and integrate to obtain*

$$C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))\varphi(x)}{|x-y|^{N+2s}} dydx = \int_\Omega f\varphi.$$

The expression on the left-hand side is well defined since $(-\Delta)^s u = f \in L^2(\Omega)$. Thanks to the dominated convergence theorem and Fubini's theorem, we can rewrite the left-hand side of the previous equation in the following ways

$$C(N,s)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{|x-y|\geq\varepsilon} \frac{(u(x)-u(y))\varphi(x)}{|x-y|^{N+2s}} dydx = C(N,s)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{|x-y|\geq\varepsilon} \frac{-(u(x)-u(y))\varphi(y)}{|x-y|^{N+2s}} dydx.$$

Thus, averaging these two expressions and using the dominated convergence theorem, again, we get

$$C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dxdy = \int_\Omega f\varphi.$$

Motivated by the previous remark we provide the following notion of weak solution for the problem (5.57).

Definition 88. *Given $f \in L^2(\Omega)$, we say that $u \in H^s_\Omega(\mathbb{R}^N)$ is a weak solution of (5.57) if*

$$\frac{C(N,s)}{2}\langle u, v\rangle_{H^s_\Omega(\mathbb{R}^N)} = \int_\Omega fv \tag{5.59}$$

for all $v \in H^s_\Omega(\mathbb{R}^N)$, where $C(N,s)$ is the normalizing constant of the fractional Laplacian.

The approach that we follow for studying the linear problem (5.57) is quite standard and is explained with more or less details in many references (see e.g. [29], [15] and the references therein). The following proposition summarizes some properties of the weak solutions of the previous problem.

Proposition 89. *For each $f \in L^2(\Omega)$ problem* (5.57) *has an unique weak solution $u \in H^s_\Omega(\mathbb{R}^N)$. Moreover, if we define the solution operator $T : L^2(\Omega) \to H^s_\Omega(\mathbb{R}^N)$, that assigns for each datum f its unique solution u, it satisfies the following properties:*

(i) *T is continuous and linear.*

(ii) *If $i : H^s_\Omega(\mathbb{R}^N) \to L^2(\Omega)$ is the inclusion operator then the compositions $T_1 := i \circ T : L^2(\Omega) \to L^2(\Omega)$ and $T_2 := T \circ i : H^s_\Omega(\mathbb{R}^N) \to H^s_\Omega(\mathbb{R}^N)$ are compact, self-adjoint and positive definite.*

Proof. The well definition of T follows directly from the Riesz representation lemma for Hilbert spaces and so do the linearity and the continuity of the operator. On the other hand, *Theorem* 68 implies that i is a linear compact operator which implies that both $i \circ T$ and $T \circ i$ are linear and compact. Hence, it only remains to prove the self-fandjointness and the fact that these two operators are positive definite. Let us prove the result for T_1, the reasoning for T_2 is completely analogous. Given $u, v \in H^s_\Omega(\mathbb{R}^N)$ and consider

$$\langle T_1(u), v\rangle_{H^s_\Omega(\mathbb{R}^N)} = \frac{2}{C(N,s)} \int_\Omega uv = \langle u, T_1(v)\rangle_{H^s_\Omega(\mathbb{R}^N)}.$$

If we make $u = v$ in the previous computation we also get that $\langle T_1(u), u\rangle_{H^s_\Omega(\mathbb{R}^N)} \geq 0$ and that $\langle T_1(u), u\rangle_{H^s_\Omega(\mathbb{R}^N)} = 0$ only when $u = 0$. □

The following theorem provides us a basic regularity result for the weak solutions of problem (5.57). For more information about regularity for weak solutions of linear problems associated to the fractional Laplacian and other nonlocal operators, see [30], [36] and the references therein.

Proposition 90. *Let Ω be a $C^{1,1}$ bounded domain in $\mathbb{R}^N$. If $f \in L^\infty(\Omega)$ then any weak solution u of* (5.57) *satisfies $u \in C^{0,s}(\Omega)$ and*

$$\|u\|_{C^{0,s}(\Omega)} \leq C\|f\|_{L^\infty(\Omega)},$$

for some constant $C > 0$ that depends only on N, s and Ω.

Proof. See [[32], Theorem 1.1] and [[29], Proposition 7.2]. □

From the definition of the fractional Laplacian it is easy to check that it distinguishes global maximizers and global minizers as discussed in *Section* 2.1. Thus, it is natural to ask whether this operator satisfies, at least, a weak version of the maximum principle for elliptic operators. The following proposition addresses this question.

Proposition 91. *If $f^- = 0$ then any weak solution u of* (5.57) *satisfies that $u^- = 0$*

Proof. In virtue of *Proposition* 86 we can use u^- as a test function in (5.59) which implies, using again the same proposition, that

$$-\frac{C(N,s)}{2}\|u^-\|^2_{H^s_\Omega(\mathbb{R}^N)} = \int_\Omega f u^- - \frac{C(N,s)}{2}\langle u^+, u^-\rangle_{H^s_\Omega(\mathbb{R}^N)} \geq 0,$$

which implies that $\|u^-\|_{H^s_\Omega(\mathbb{R}^N)} = 0$. □

The previous proposition can be generalized using suitable definitions for weak subsolutions and weak supersolutions. Also this analysis can be extended for other nonlocal operators with nonhomogeneus Dirichlet datum on the nonlocal boundary, see [29].

5.2 Spectrum of the fractional Laplacian

Eventhough the spectral fractional Laplacian presented in *Section* 2.2 was introduced to motivate the idea of computing fractional derivatives, this operator, in general, differs from the integro-differential operator given by $(-\Delta)^s$ essentially because the former arises through a local process and the latter is purely nonlocal. We will precise this ideas in this section.

For some purposes that will be precised in the next chapter we will study the spectrum of the fractional Laplacian including some suitable type of weights.

Definition 92. *Given $h \in L^\infty(\Omega)$, $u \in H^s_\Omega(\mathbb{R}^N) \setminus \{0\}$ and $\lambda \in \mathbb{R}$ we say that (u, λ) is weak pair eigenfunction-eigenvalue associated to $(-\Delta)^s$ in Ω, with weight h and with Dirichlet condition if*

$$\frac{C(N,s)}{2}\langle u, v\rangle_{H^s_\Omega(\mathbb{R}^N)} = \lambda \int_\Omega huv, \tag{5.60}$$

for all $v \in H^s_\Omega(\mathbb{R}^N)$.

Analogously to the case of the Laplacian, the eigenvalue problem for $(-\Delta)^s$ can be approached through the inverse or solution operator associated to the fractional Laplacian which was studied in the previous section.

Remark 93. *Notice that, in this case, for any datum $f \in L^2(\Omega)$ the solution of the associated linear problem with the weight $h \in L^\infty(\Omega)$ has the form $T(hf)$. In this case we can define the operators $T_{h_1} : H^s_\Omega(\mathbb{R}^N) \to H^s_\Omega(\mathbb{R}^N)$ and $T_{h_2} : L^2(\Omega) \to L^2(\Omega)$ by T_1 and $T2$ composed with multiplication by h. These two operators are also linear, compact and self-adjoint but, in general, are not positive unless h is positive.*

Proposition 89 and *Remark* 93 showed us that the "inverse weighted fractional Laplacian" T_{h_1} (resp. T_{h_2}) satisfies the hypotheses of the spectral theorem [[1], Theorem 4.27] which implies that, for some types of h, both $L^2(\Omega)$ and $H^s_\Omega(\mathbb{R}^N)$ have Hilbert bases generated by the eigenfunctions of these operators (that are essentially the same). Moreover, the eigenvalues of both of them coincide, have finite multiplicity and are, in fact, the inverses of the eigenvalues of the fractional Laplacian with weight h. All these results are fairly standard (see [8] for a general scheme in the local case and [[34], Proposition 9] for a partial proof in the non local case). Nonetheless, we will summarize them, and other important results that will be required below in the following propositions.

Proposition 94. *If $h = 1$ all the weak pairs eigenfunction-eigenvalues associated to $(-\Delta)^s$ in Ω with Dirichlet condition are given by the sequence $\{(\varphi_n, \lambda_n(s))\}_{n\in\mathbb{N}}$ that satisfies*

(i) $\{\lambda_n(s)\}_{n\in\mathbb{N}}$ *is a nondecreasing sequence of positive numbers, each* $\lambda_{n(s)}$ *has finite multiplicity and*

$$\lim_{n\to\infty} \lambda_n(s) = \infty.$$

(ii) $\{\varphi_n\}_{n\in\mathbb{N}}$ *constitutes an orthonormal basis for* $L^2(\Omega)$ *and, up to normalization, constitutes a orthonormal basis for* $H^s_\Omega(\mathbb{R}^N)$.

The following theorem provides a regularity result about the eigenfunctions of the fractional Laplacian and, in particular, shows that these functions cannot coincide with the ones obtained from the spectral analysis of the Laplacian since the latter are smooth.

Theorem 95. *Every weak eigenfunction* u *associated to* $(-\Delta)^s$ *in* Ω*, with weight* $h = 1$ *and with Dirichlet boundary condition satisfies that* $u \in C^{0,s}(\mathbb{R}^N)$*. Moreover, this result is sharp.*

Proof. See [35]. □

The next proves that the first positive eigenvalue of the problem (5.60) for any suitable weight $h \in L^\infty(\Omega)$ is simple which provides an analogous of the result for the Laplacian due to T. Kato and P. Hess (see [19]). There already exist proofs of this result when h is constant (see [34]). In our case the ideas are inspired by [10].

Proposition 96. *If* $h > 0$ *in a subset of* Ω *with positive measure, then*

(i) *There exists a least positive eigenvalue* $\lambda_1(h,s)$ *of* (5.60).

(ii) *Every eigenfunction* φ *associated to* $\lambda_1(h,s)$ *is of one sign, i.e. either* $\varphi^+ = 0$ *or* $\varphi^- = 0$ *a.e. in* Ω.

(iii) $\lambda_1(h,s)$ *is simple, i.e. the dimension of the eigenspace associated to* $\lambda_1(h,s)$ *is 1.*

Proof. Notice that since $h > 0$ in some set with positive measure we can take a function φ in $C_0^\infty(\Omega)$ approaching in $L^2(\Omega)$ the characteristic function of the set where $h > 0$, so that $\int_\Omega h\varphi^2 > 0$. Hence, by the variational characterization of the eigenvalues of T_{h_1} (see [[8], Theorem 1.3.4]) we get that

$$\frac{1}{\lambda_1(h,s)} = \sup_{u\in H^s_\Omega(\mathbb{R}^N)\setminus\{0\}} \frac{\int_\Omega hu^2}{\|u\|^2_{H^s_\Omega(\mathbb{R}^N)}} > 0. \tag{5.61}$$

Also notice that, by *Proposition* 86 $\||u|\|^2_{H^s_\Omega(\mathbb{R}^N)} \leq \|u\|^2_{H^s_\Omega(\mathbb{R}^N)}$ and since the L^2-norm of both functions is the same, $|u|$ induce a value bigger or equal than u in the Rayleigh quotient of the right hand side of (5.61). In fact, the same proposition guarantees that Rayleigh quotient coincides if and only if either u^+ or u^- is 0 almost everywhere. On the other hand, if we take u and v eigenfunctions associated with $\lambda_1(h,s)$, we can define the sets

$$A := \{t \in \mathbb{R} | u + tv \geq 0, \quad a.e. \text{ in } \quad \Omega\}, \qquad B := \{t \in \mathbb{R} | u + tv \leq 0 \quad a.e. \text{ in } \quad \Omega\}.$$

It follows immediatly that $A \cup B = \mathbb{R}$. Also, it is not difficult to see that since $u, v \in L^1(\Omega)$ there exist t_1 and t_2 such that the sets $\{x \in \Omega | u + t_1 v \geq 0\}$ and $\{x \in \Omega | u(x) + t_2 v(x) \leq 0\}$ have positive measure, implying, by the previous part of the proof, that A and B are nonempty. Lastly, let us see that A is closed (the analysis for B is completely analogous). Given any sequence $\{t_n\}_{n\in\mathbb{N}}$ in A we can define the sets $A_n := \{x \in \Omega | u(x) + t_n v(x) \geq 0\}$. Since $|A_n^c| = 0$ for each $n \in \mathbb{N}$ it follows that $\left|\bigcup_{n\in\mathbb{N}} A_n^c\right| = 0$, hence for almost every $x \in \Omega$ we have that $u(x) + tv(x) \geq 0$. Finally, by connectedness we get that A and B cannot be disjoint implying that u and v must be linear dependent. □

Using the variational characterization of the eigenvalues (see [8]) it is possible to deduce some Poincaré-type inequalities that will be useful in our variational analysis for the semilinear problems.

Proposition 97. *The optimal constant of the Poincaré inequality* (5.58) *is given by* $\frac{1}{\lambda_1(s)^{1/2}}$. *Moreover, if* $\{\varphi_n\}_{n\in\mathbb{N}}$ *is the sequence of weak eigenfunctions of the fractional Laplacian of Proposition 94 we have that for* $k \geq 2$

$$\|v\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda_k(s)} \|v\|^2_{H^s_\Omega(\mathbb{R}^N)}, \quad \forall v \in \langle \varphi_1, \varphi_2, \cdots, \varphi_{k-1}\rangle^\perp, \tag{5.62}$$

$$\|v\|^2_{L^2(\Omega)} \geq \frac{1}{\lambda_{k-1}(s)} \|v\|^2_{H^s_\Omega(\mathbb{R}^N)}, \quad \forall v \in \langle \varphi_1, \varphi_2, \cdots, \varphi_{k-1}\rangle \tag{5.63}$$

Proof. The fact that $\frac{1}{\lambda_1(s)^{1/2}}$ provides the optimal constant in the Poincaré inequality follows immediatly from the variational characterization (5.61). In the same spirit, we have that for any self-adjoint, compact and positive operator like $i \circ T$, its eigenvalues have the following variational characterization (see [[8], Section 1.3])

$$\frac{1}{\lambda_k(s)} = \sup_{v\in\langle u_1,\cdots,u_{k-1}\rangle^\perp\setminus\{0\}} \frac{\int_\Omega v^2}{\|v\|^2_{H^s_\Omega(\mathbb{R}^N)}} = \inf_{v\in\langle u_1,\cdots,u_k\rangle\setminus\{0\}} \frac{\int_\Omega v^2}{\|v\|^2_{H^s_\Omega(\mathbb{R}^N)}},$$

thus, from this characterization the other part of the proof follows directly. □

6. A semilinear problem

In this chapter we are interested in studying a generalization of the linear problem addressed in the previous chapter. Namely,

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega^c, \end{cases} \tag{6.64}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is an open bounded set, $s \in (0,1)$ and $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function. Specifically, we are interested in the solutions of (6.64) that come from a variational setting, i.e. we are looking for weak solutions. In order to give a variational formulation to our problem it is necessary to impose, from now on, some extra conditions on f that guarantee the integrability of the right hand side of the equation:

(f_1') There exist $a > 0, b \in \mathbb{R}$ and $p \in [2, 2^*)$ such that $|f(t)| \leq a|t|^{p-1} + b$.

Here, as in the previous chapters, $2^* := \dfrac{2N}{N-2s}$ is the fractional Sobolev critical exponent.

Analogously to the motivation provided in *Remark* 87 we provide the following notion of weak solution for the problem (6.64).

Definition 98. *We say that $u \in H^s_\Omega(\mathbb{R}^N)$ is a weak solution of* (6.64) *if*

$$\frac{C(N,s)}{2} \langle u, v \rangle_{H^s_\Omega(\mathbb{R}^N)} = \int_\Omega f(u) v \tag{6.65}$$

for all $v \in H^s_\Omega(\mathbb{R}^N)$, where $C(N,s)$ is the normalizing constant of the fractional Laplacian.

Actually, as stated by the following theorem, under certain assumptions on f it is possible to guarantee that a weak solution, in the sense of the previous definition, has some regularity.

Theorem 99. *If Ω is of class $C^{1,1}$ and f is locally Lipschitz then every weak solution u of* (6.64) *satisfies that $u \in C^{0,s}(\mathbb{R}^N)$.*

Proof. See [[31], Theorem 1.4]. □

The following result shows us that, in fact, we can translate the problem of finding weak solutions into the problem of finding critical points for a suitable energy functional and also shows that the derivative of that functional has an useful structure of the form "identity minus compact".

Proposition 100. *Let $F : \mathbb{R} \to \mathbb{R}$ be defined as $F(t) := \int_0^t f(s) ds$.*

1. *The functional*

$$J : H^s_\Omega(\mathbb{R}^N) \to \mathbb{R},$$
$$u \to \frac{C(N,s)}{4} \|u\|^2_{H^s_\Omega(\mathbb{R}^N)} - \int_\Omega F(u),$$

is of class C^1 and satisfies

$$DJ(u)(v) = \frac{C(N,s)}{2} \langle u,v \rangle_{H^s_\Omega(\mathbb{R}^N)} - \int_\Omega f(u)v, \qquad u,v \in H^s_\Omega(\mathbb{R}^N).$$

2. *The functional*

$$K : H^s_\Omega(\mathbb{R}^N) \to \left(H^s_\Omega(\mathbb{R}^N)\right)^*,$$
$$u \to \int_\Omega f(u)(\cdot),$$

is compact.

Proof. 1. The differentiability of the first term of J readily follows from noticing that it comes from a continuous bilinear form, namely

$$b : H^s_\Omega(\mathbb{R}^N) \times H^s_\Omega(\mathbb{R}^N) \to \mathbb{R},$$
$$(u,v) \to \frac{C(N,s)}{4} \langle u,v \rangle_{H^s_\Omega(\mathbb{R}^N)}.$$

On the other hand, for the differentiability of the second term of J we have that for u and v in $H^s_\Omega(\mathbb{R}^N)$

$$|\int_\Omega F(u+v) - F(u) - f(u)v| \le \int_\Omega \int_0^1 |(f(u+tv) - f(u))v| dt.$$

Applying Fubini's theorem and Holder's inequality with exponents 2^* and its conjugate, namely c, we get

$$\left| \int_\Omega F(u+v) - F(u) - f(u)v \right| \le \int_0^1 \|f(u+tv) - f(u)\|_{L^c(\Omega)} \|v\|_{L^{2^*}(\Omega)} dt.$$

Thus, if we apply the Sobolev inequality (*Corollary* 67), the dominated convergence theorem and Vainberg's lemma [1] it follows that

$$\frac{|\int_\Omega F(u+v) - F(u) - f(u)v|}{\|v\|_{H^s_\Omega(\mathbb{R}^N)}} \le \int_0^1 \|f(u+tv) - f(u)\|_{L^c(\Omega)} dt \to 0,$$

as $v \to 0$ in $H^s_\Omega(\mathbb{R}^N)$. The continuity of DJ follows directly from applying Vainberg's lemma.

2. Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^s_\Omega(\mathbb{R}^N)$ and let $v \in H^s_\Omega(\mathbb{R}^N)$ with unitary norm. *Theorem* 68 implies that, up to a subsequence, $u_n \to u$ for some $u \in L^{p-1}(\Omega)$, with

[1]**Vainberg's lemma:** If $f : \mathbb{R} \to \mathbb{R}$ satisfies (f_1') then $N : L^{2^*}(\Omega) \to L^{\frac{2^*}{p-1}}(\Omega)$ given by $N(u) := f(u)$ is a well defined, bounded and continuous operator. See [[24], Theorem 2, Chapter 3]

p given in the condition (f_1'). On the other hand, since $H^s_\Omega(\mathbb{R}^N)$ is reflexive, $\{u_n\}_{n\in\mathbb{N}}$ also converges, up to a subsequence, weakly to u in $H^s_\Omega(\mathbb{R}^N)$. In addition

$$|I(u_n)(v) - I(u)(v)| \leq C\|f(u_n) - f(u)\|_{L^{p^*}(\Omega)},$$

where p^* is the conjugate exponent of p. Finally, the result follows from applying Vainberg's lemma .

□

Since our problem relates a linear part given by the integro-differential operator and a nonlinear component given by f, it is customary (and necessary) to understand the behaviour of f in terms of the fundamental frequencies of the linear part, i.e. its spectrum. Given the results of the previous chapter we are in position to specify a set of hypotheses on f that allows us to approach the weak problem associated with (6.64). From now on we assume

(f_1) $f \in C^1(\mathbb{R},\mathbb{R})$ and $f(0) = 0$,

(f_2) $\dfrac{2f'(0)}{C(N,s)} < \lambda_1(s)$,

(f_3) There exist $a > 0, b \in \mathbb{R}$ and $p \in [2, 2^*)$ such that $|f'(t)| \leq a|t|^{p-2} + b$,

(f_4) There exists $k \geq 2$ such that

$$\frac{2f'(\infty)}{C(N,s)} := \lim_{|t|\to\infty} \frac{2f(t)}{C(N,s)t} \in \left(\lambda_k(s), \lambda_{(k+1)}\right)(s). \tag{6.66}$$

(f_5) There exists a constant $\gamma < \lambda_{k+1}(s)$ such that $f'(t) \leq \frac{C(N,s)}{2}\gamma$ for every $t \in \mathbb{R}$.

Under these hypotheses we prove the following multiplicity results which give a first step in extending the main result of [4] to the nonlocal case.

Theorem 101. *If f satisfies the hypotheses (f_1)–(f_4) then the weak problem associated with (6.64) has at least 2 nontrivial continuous solutions ω_1 and ω_2. Moreover, these solutions satisfy that ω_1 is nonnegative and ω_2 is nonpositive on Ω.*

Theorem 102. *If f satisfies the hypotheses $(f_1)-(f_5)$ then the weak problem associated with (6.64) has at least 2 nontrivial continuous solutions ω_3 and ω_4. Moreover, ω_3 has the variational characterization*

$$J(\omega_3) = \max_{x\in\langle\varphi_1,\cdots,\varphi_k\rangle} \min_{y\in\langle\varphi_{k+1},\cdots\rangle} J(x+y),$$

where $\{\varphi_n\}_{n\in\mathbb{N}}$ is defined as in Proposition 94.

Remark 103. *The solutions ω_1 and ω_2 in principle are not necessarily different from ω_3 and ω_4. The next step of this analysis (that is a work still in progress) would consist in providing arguments to distinguish all the four solutions.*

In order to prove these results we will prove several lemmas that will show the existence of each one of the solutions separately. In order to prove the existence of ω_1 and ω_2 it is necessary to introduce the following notation

$$f^+(t) := \begin{cases} f(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and

$$f^-(t) := \begin{cases} f(t), & t \leq 0, \\ 0, & t > 0. \end{cases}$$

Also, we can define the functionals J^+ and J^- changing in the definition of J the function f by f^+ and f^-, respectively.

The proof of the existence of ω_1 and ω_2 relies on minmax techniques. The following definition and theorem provide the required machinery.

Definition 104. *Let X be a real Banach space and let $J : X \to \mathbb{R}$ be a C^1 functional.*

1. *We say that $\{u_n\}_{n\in\mathbb{N}}$ is a* ***Palais-Smale sequence*** *or (PS)-sequence for J if $\{J(u_n)\}_{n\in\mathbb{N}}$ is bounded and $J(u_n) \to 0$ in X^*.*

2. *We say that J satisfies the* ***Palais-Smale condition*** *or (PS)-condition if any (PS)-sequence for J has a convergent subsequence.*

Theorem 105. *[Mountain Pass Theorem] Let X be a real Banach space and let $J : X \to \mathbb{R}$ be a C^1 functional such that*

1. *$J(0) = 0$,*

2. *There exist $R > 0$ and $c > 0$ such that $J(y) \geq c$ for every $y \in \partial B_R(0)$,*

3. *There exists $x \in B_R(0)^c$ such that $J(x) \leq 0$,*

4. *J satisfies the (PS)-condtion.*

Then J has a critical point u that satisfies $J(u) \geq c$.

Proof. See [[28],Theorem 2.2]. □

Given that our strategy to prove the existence of ω_1 and ω_2 consists in applying the Mountain Pass Theorem to some suitable functionals, we will require to check the (PS)-condition. The following lemma gives us an alternative way to check this condition for certain type of functionals.

Lemma 106. *Let H be a real Hilbert space and let $J : H \to \mathbb{R}$ be a C^1 functional. Suppose that $DJ = cR - K$ where $c \in \mathbb{R} \setminus \{0\}$, $R : H \to H^*$ is the Riesz mapping and $K : H \to H^*$ is a (nonlinear) compact operator. Then every bounded (PS)-sequence for J has a convergent subsequence.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded (PS)-sequence for J. Hence, since K is compact $\{K(u_n)\}_{n\in\mathbb{N}}$ converges, up to a subsequence. Therefore, since $\{DJ(u_n)\}_{n\in\mathbb{N}}$ is also convergent we get that $\{R(u_n)\} \to v$ as $n \to \infty$, up to a subsequence, for some $v \in H^*$. The result follows from the fact that R is a bijective isometry. □

Now we are in position to prove *Theorem* 101.

Proof of Theorem 101. Let us prove the result for J^+, the reasoning is completely analogous for J^-. Notice that since f satisfies (f_1') then f^+ and f^- do too. Implying that we can associate a energy functional to each one. Moreover, the derivative of both functionals are of the form "identity minus compact". Hence, in virtue of *Proposition* 100 and *Lemma* 106, it suffices to check that any (PS)-sequence is bounded. Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS)-sequence for J^+. For n large we have that

$$\begin{aligned}\|u_n^-\|_{H^s_\Omega(\mathbb{R}^N)} \geq DJ^+(u_n)(-u_n^-) &= -\frac{C(N,s)}{2}\langle u_n, u_n^-\rangle_{H^s_\Omega(\mathbb{R}^N)} + \int_\Omega f^+(u_n)u_n^- \\ &= -\frac{C(N,s)}{2}\langle u_n, u_n^-\rangle_{H^s_\Omega(\mathbb{R}^N)},\end{aligned}$$

which implies that

$$\frac{C(N,s)}{2}\|u_n^-\|^2_{H^s_\Omega(\mathbb{R}^N)} \leq \|u_n^-\|_{H^s_\Omega(\mathbb{R}^N)} + \frac{C(N,s)}{2}\langle u_n^+, u_n^-\rangle_{H^s_\Omega(\mathbb{R}^N)}.$$

Since $\langle u_n^+, u_n^-\rangle_{H^s_\Omega(\mathbb{R}^N)} \leq 0$ this inequality proves that $\{(u_n)^-\}_{n\in\mathbb{N}}$ is bounded in $H^s_\Omega(\mathbb{R}^N)$ by a constant $C > 0$. Let $(\varphi_n, \lambda_n(s))$ be as in *Proposition* 94 and let us consider the orthogonal decomposition $u_n^+ = v_n + w_n$ with $v_n \in \langle\varphi_1, \cdots, \varphi_k\rangle$ and $w_n \in \langle\varphi_1, \cdots, \varphi_k\rangle^\perp$. Since $\{u_n\}_{n\in\mathbb{N}}$ is a (PS)-sequence for J, for n large we have that

$$\|w_n + v_n\|_{H^s_\Omega(\mathbb{R}^N)} \geq DJ^+(u_n)(w_n - v_n) = \frac{C(N,s)}{2}\langle u_n, w_n - v_n\rangle_{H^s_\Omega(\mathbb{R}^N)} - \int_\Omega f(u_n^+)(w_n - v_n).$$

Using orthogonality, the boundedness of $\{(u_n)^-\}_{n\in\mathbb{N}}$ in $H^s_\Omega(\mathbb{R}^N)$, and Cauchy-Schwartz inequality, we get

$$\frac{C(N,s)}{2}\left(\|w_n\|^2_{H^s_\Omega(\mathbb{R}^N)} - \|v_n\|^2_{H^s_\Omega(\mathbb{R}^N)}\right) - \int_\Omega f(u_n^+)(w_n - v_n) \leq C\|w_n + v_n\|_{H^s_\Omega(\mathbb{R}^N)}.$$

Given that f is asymptotically linear (condition (f_4)) there exists a continuous function h such that $f(t) = f'(\infty)t + h(t)$ with $h(t) = o(t)$ as $|t| \to \infty$. This consideration implies that

$$\begin{aligned}\frac{C(N,s)}{2}\left(\|w_n\|^2_{H^s_\Omega(\mathbb{R}^N)} - \frac{2f'(\infty)}{v}\int_\Omega w_n^2\right) + \frac{C(N,s)}{2}\left(\frac{2f'(\infty)}{C(N,s)}\int_\Omega v_n^2 - \|v_n\|^2_{H^1(\Omega)}\right) \\ \leq \int_\Omega h(u_n^+)(w_n - v_n) + C\|w_n + v_n\|_{H^s_\Omega(\mathbb{R}^N)}.\end{aligned}$$

A direct application of *Proposition* 97 implies that there exists a constant $K > 0$ such that

$$K\left(\|w_n\|^2_{H^s_\Omega(\mathbb{R}^N)} + \|v_n\|^2_{H^s_\Omega(\mathbb{R}^N)}\right) \leq \|w_n + v_n\|_{H^s_\Omega(\mathbb{R}^N)} + \int_\Omega h(w_n + v_n)(w_n - v_n).$$

On the other hand, the properties of h imply that there exists $a>0$ such that $|h(t)| \le \frac{K}{2\lambda_1(s)}|t|+a$, where K is the constant of the previous inequality. Therefore, from the Cauchy-Schwartz inequality, Poincaré inequality (*Theorem* 84) and by orthogonality it follows that

$$\begin{aligned}\int_\Omega h(w_n+v_n)(w_n-v_n) &\le \frac{1}{\lambda_1(s)^{\frac{1}{2}}}\|w_n+v_n\|_{H^s_\Omega(\mathbb{R}^N)}\|h(w_n+v_n)\|_{L^2(\Omega)} \\ &\le \frac{K}{2\lambda_1(s)}\|w_n+v_n\|^2_{H^s_\Omega(\mathbb{R}^N)} + \frac{a}{\lambda_1(s)^{\frac{1}{2}}}|\Omega|^{\frac{1}{2}}\|w_n+v_n\|_{H^s_\Omega(\mathbb{R}^N)}.\end{aligned}$$

Finally, combining the last two inequalities we get that our (PS)-sequence is bounded. The next part of the proof consists in showing that J^+ has the "mountain pass geometry around zero" (hypothesis 2 in *Theorem* 105). First of all, notice that we can apply Taylor's theorem to F to get that for some $\delta>0$ and $|t|<\delta$ it holds

$$F(t) = \frac{f'(0)}{2}t^2 + o(t^2) = \left(\frac{f'(0)}{2} + \frac{o(t^2)}{t^2}\right)t^2.$$

Therefore, there exists $0<\delta'<\delta$ such that for $|t|<\delta'$ we have $F^+(t) \le \frac{c}{2}t^2$ for some $c \in \left(0, \frac{C(N,s)f'(0)}{2}\right)$. Also, our assumptions implies that $\frac{f^+(t)}{t}$ is bounded, thus there exists a constant $c'>c$ such that $F^+(t) \le \frac{c'}{2}t^2$ for every $t\in\mathbb{R}$.

Keeping in mind these considerations we can estimate $J^+(u)$ for u small in $H^s_\Omega(\mathbb{R}^N)$ in the following way

$$\begin{aligned}J^+(u) \ge \frac{C(N,s)}{4}\|u\|^2_{H^s_\Omega(\mathbb{R}^N)} - \frac{c}{2}\int_{\{|u|<\delta'\}}u^2 - \frac{c'}{2}\int_{\{|u|\ge\delta'\}}u^2 = \frac{C(N,s)}{4}\|u\|^2_{H^s_\Omega(\mathbb{R}^N)} - \frac{c}{2}\int_\Omega u^2 \\ - \frac{c'-c}{2}\int_{\{|u|\ge\delta'\}}u^2.\end{aligned}$$

Using the critical continuous embedding, Poincaré inequality, Holder inequality with $p=\frac{2^*}{2}$ and $q=\frac{2^*}{2^*-2}$ and Chebyshev inequality it follows that

$$\int_{\{|u|\ge\delta'\}}u^2 \le C_N\|u\|^2_{H^s_\Omega(\mathbb{R}^N)}\left(\frac{\|u\|_{L^2(\Omega)}}{\delta'^2}\right)^{\frac{1}{q}}.$$

Thus, there exists a positive constant $C>0$ and $\varepsilon>0$ such that for $\|u\|_{H^s_\Omega(\mathbb{R}^N)} \le \varepsilon$

$$J^+(u) \ge C\|u\|^2_{H^s_\Omega(\mathbb{R}^N)}.$$

Now, to check hypothesis 3 of the Mountain Pass Theorem, let us notice that since $f'(\infty) > \dfrac{C(N,s)\lambda_1(s)}{2}$ there exist constants $a > \dfrac{C(N,s)\lambda_1(s)}{2}$ and b such that, for $t\in\mathbb{R}$, $F^+(t) > \frac{a}{2}t^2+b$. Hence, let us notice that the sequence of multiples of a nonzero eigenfunction φ_1 associated to $\lambda_1(s)$ given by $\{n\varphi_1\}_{n\in\mathbb{N}}$ satisfies

$$J^+(nu_1) = \frac{C(N,s)}{4}\|nu_1\|^2_{H^s_\Omega(\mathbb{R}^N)} - \int_\Omega F^+(nu_1)$$
$$\leq \frac{C(N,s)}{4}\|nu_1\|^2_{H^s_\Omega(\mathbb{R}^N)} - \frac{a}{2}\|nu\|^2_{L^2(\Omega)} - |\Omega|b \to -\infty, \quad n\to\infty,$$

in virtue of Poincaré inequality. Hence, the mountain pass theorem guarantees that J^+ has a critical point ω_1 such that $J^+(\omega_1) > 0$, implying that ω_1 is not identically 0.

We claim that $\omega_1^- = 0$ a.e. in $\mathbb{R}^N$. Indeed

$$0 = DJ^+(\omega_1)(\omega_1^-) = \frac{C(N,s)}{2}\langle\omega_1,\omega_1^-\rangle_{H^s_\Omega(\mathbb{R}^N)},$$

thus

$$0 \leq \langle\omega_1^-,\omega_1^-\rangle_{H^s_\Omega(\mathbb{R}^N)} = \langle\omega_1^+,\omega_1^-\rangle_{H^s_\Omega(\mathbb{R}^N)} \leq 0.$$

Finally, since ω_1 is also a critical point of J and by *Theorem* 99 it follows that ω_1 satisfies the desired properties.

□

In order to distinguish these two solutions from the others it is necessary to obtain some extra qualitative information of them. In our case, this information is obtained through topological tools that will be briefly described in the following definitions and results. All the ideas in the sequel will envolve the assumption that the critical points are isolated. Notice that, in general, if one critical point of J is not isolated then there exist infinitely many, which makes trivial our multiplicity result. Moreover, if the critical points of J^+ or the critical points of J^- are not isolated we get infinitely many nonnegative and nonpositive solutions of our problem. A very important topological tool that will be used in this work is the *Leray-Schauder topological degree*, see [21].

Let X be a real Banach space and let $O \subset X$ be a bounded subset. Given a functional $T : \overline{O} \to X$ such that $T = I - K$ where $K : \overline{\Omega} \to X$ is a compact (nonlinear) operator then we denote the degree of T at any point $b \notin T(\partial O)$ by $d(T,O,b)$
The following lemmas summarize the properties of the degree that will be required in the sequel.

Lemma 107. *Let O be a bounded subset of a be a real Banach space X and let $T = I - K : \overline{O} \to X$ where $K : \overline{O} \to X$ is a compact (nonlinear) operator. Then for every $b \notin T(\partial O)$ we have that*

- ***Excision:*** *If $\bigcup_{l\in L} O_l \subset O$ is an union of pairwise disjoint open sets such that $T^{-1}(b) \in \bigcup_{l\in L}\Omega_l$ then*
$$d(T,O,b) = \sum_{l\in L} d(T,O_l,b)$$

- ***Homotopy invariance:*** *Let $H \in C(\overline{O} \times [0,1], H)$ be defined by $H(x,t) = x - S(x,t)$ with $S(\cdot,t)$ a compact operator for every $t \in [0,1]$. If $b \notin H(\partial O \times [0,1])$, then $d(H(\cdot,t), O, b)$ is independent of t.*

- *If K is differentiable at the origin, $K(0) = 0$ and 1 is not a characteristic value of $DK(0)$ then we have that*
$$d(T, B_\rho(0), 0) = (-1)^\beta,$$
where $\rho > 0$ is such that $B_\rho(0)$ does not contain any other root of T and β is the sum of the algebraic multiplicities of the negative eigenvalues of $T'(0)$.

Proof. See chapters 2 and 3 in [21]. □

In our particular case, where we are dealing with a functional $J \in C^1(H,\mathbb{R})$ such that $DJ : H \to H^*$ we can use Riesz representation lemma to get an element in H that represents $DJ(u)$ for every value of $u \in H$. It is costumary to denote this element by $\nabla J(u)$. Hence, in virtue of *Proposition* 100, the functional $\nabla J : H \to H$ satisfies the conditions to compute its degree with respect some types of values and domains. Now, we introduce a version of the so-called *Lyapunov-Schmidt reduction method* which will be used to obtain the third solution, namely ω_3. Under appropriate conditions this technique provides a general procedure to transform a variational infinite-dimensional problem into an equivalent (often easy-to-solve) finite-dimensional problem. Here we follow [5], [4] and the references therein.

Theorem 108. *Let H be a real Hilbert space. Let X and Y be closed subspaces of H such that $H = X \oplus Y$. Assume that $J : H \to \mathbb{R}$ is a functional of class C^1. If there is a constant $m > 0$ such that*

$$\langle \nabla J(x+y_1) - \nabla J(x+y_2), y_1 - y_2 \rangle \geq m\|y_1 - y_2\|^2 \quad \text{for all} \quad x \in X,\ y_1, y_2 \in Y, \tag{6.67}$$

then:

(i) there exists a continuous function $\psi : X \to Y$ such that
$$J\big(x+\psi(x)\big) = \min_{y\in Y} J(x+y).$$

(ii) The function
$$\begin{aligned} \tilde{J} : X &\longrightarrow \mathbb{R} \\ x &\longrightarrow \tilde{J}(x) := J\big(x+\psi(x)\big) \end{aligned}$$
is of class C^1, and
$$\big\langle \nabla \tilde{J}(x_1), x_2 \big\rangle = \big\langle \nabla J\big(x_1 + \psi(x_1)\big), x_2 \big\rangle \quad \text{for all} \quad x_1, x_2 \in X.$$
Moreover, $\psi(x) \in Y$ is the unique element satisfying
$$\big\langle \nabla J\big(x+\psi(x)\big), y \big\rangle = 0 \quad \text{for all} \quad y \in Y.$$

(iii) *An element $x_0 \in X$ is a critical point of $\widetilde{J}$ if and only if $u_0 := x_0 + \psi(x_0)$ is a critical point of J.*

(iv) *Let $S \subset X$ and $\Sigma \subset H$ be open bounded sets such that*

$$\{x+\psi(x); x \in S\} = \Sigma \cap \{x+\psi(x); x \in X\}.$$

If $\nabla \widetilde{J}(x) \neq 0$ for every $x \in \partial S$ then

$$d(\nabla J, \Sigma, 0) = d(\nabla \widetilde{J}, S, 0).$$

Thus, the strategy consists in proving that, under our hypotheses, J satisfies the assumptions of *Theorem* 108 and that functional $\widetilde{J}$ has an absolute maximizer. In this order of ideas let X be the vector space spanned by $\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$ and $Y := \langle \varphi_{k+1}, \cdots \rangle = X^{\perp}$.

Lemma 109. *The functional $J : H^s_\Omega(\mathbb{R}^N) \to \mathbb{R}$ satisfies the assumptions of Theorem 108 and it has a critical point ω_3 which if isolated satisfies that*

$$d(\nabla J, B_\rho(\omega_3), 0) = (-1)^k,$$

for some $\rho > 0$ such that $B_\rho(\omega_3)$ does not contain any other critical point of J.

Proof. Fixing $x \in X$ and $y_1, y_2 \in Y$ we have, by a direct application of the mean value theorem and the fact that f' is bounded by above,

$$\begin{aligned}\langle \nabla J(x+y_1) - \nabla J(x+y_2), y_1 - y_2\rangle_{H^s_\Omega(\mathbb{R}^N)} &= \frac{C(N,s)}{2}\|y_1 - y_2\|^2_{H^s_\Omega(\mathbb{R}^N)} - \int_\Omega (f(x+y_1) - f(x+y_2))(y_1 - y_2) \\ &\geq \frac{C(N,s)}{2}\|y_1 - y_2\|^2_{H^s_\Omega(\mathbb{R}^N)} - \gamma\|y_1 - y_2\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus, *Proposition* 97 implies that (6.67) is satisfied with $m = \frac{C(N,s)}{2} - \frac{\gamma}{\lambda_{k+1}(s)} > 0$.

Since $\frac{C(N,s)}{2} f'(\infty) \in (\lambda_k(s), \lambda_{k+1}(s))$, we can assume that there exist constants $\zeta \in \mathbb{R}$ and $\beta > \frac{2\lambda_k(s)}{C(N,S)}$ such that

$$F(t) \geq \frac{\beta}{2} t^2 + \zeta \quad \forall t \in \mathbb{R}.$$

Hence,

$$J(u) = \frac{C(N,s)}{4}\|u\|^2_{H^s_\Omega(\mathbb{R}^N)} - \int_\Omega F(u) \leq \frac{C(N,s)}{4}\|u\|^2_{H^s_\Omega(\mathbb{R}^N)} - \frac{\beta}{2}\|u\|^2_{L^2(\Omega)} - \zeta|\Omega|.$$

Using again the inequalities of *Proposition* 97, we have that for $u \in X$

$$J(u) \leq \left(\frac{C(N,s)}{4} - \frac{\beta}{2\lambda_k(s)}\right)\|u\|^2_{H^s_\Omega(\mathbb{R}^N)} - \zeta|\Omega| \to -\infty,$$

as $\|u\|_{H^s_\Omega(\mathbb{R}^N)} \to \infty$. Moreover, since $\widetilde{J}(u) \leq J(u)$ we get $\widetilde{J}(u) \to -\infty$ as $\|u\|_{H^s_\Omega(\mathbb{R}^N)} \to \infty$. Thus, by combining the latter inequality with the condition $\dim X < \infty$, we deduce the existence of some element $x_0 \in X$ satisfying

$$\widetilde{J}(x_0) = \max_{x\in X} \widetilde{J}(x) = \max_{x\in X} J\left(x+\psi(u)\right) = \max_{x\in\langle\varphi_1,\cdots,\varphi_k\rangle} \min_{y\in\langle\varphi_{k+1},\cdots\rangle} J(x+y). \tag{6.68}$$

Taking $\omega_3 = x_0+\psi(x_0)$, we see that ω_3 is a critical point of J. The fact that $d(\nabla J, B_\rho(\omega_3), 0) = (-1)^k$ for some $\rho > 0$ follows from the fact that x_0 is a global maximizer in a subspace of dimension k, properties of Leray-Schauder degree (see [6]) and (iv) in *Theorem* 108.

□

The last step towards the proof of *Theorem* 102 can be summarized as a standard application of the excision property of Leray-Schauder degree. Basically we are going to compute the "global degree" and show that the addition of the local degrees of the two solutions obtained so far (ω_3 and 0) does not match the global degree implying the existence of a second nontrival solution.

Lemma 110. *If we define the linear homotopy* $h(\lambda,t) := \lambda f'(\infty)t + (1-\lambda)f(t)$*, for each* $(\lambda,t) \in [0,1]\times\mathbb{R}$*, then there exists* $R>0$ *such that all the critical points of the energy functional* J_λ *associated to* $h(\lambda,\cdot)$ *belong to* $B_R(0)$*, for every* $\lambda \in [0,1]$*. Moreover* $d(\nabla J, B_R(0), 0) = (-1)^k$.

Proof. Let us suppose, by contradiction, that there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ of critical points of J_{λ_n}, with $\lambda_n \in [0,1]$, such that $\|u_n\|_{H^s_\Omega(\mathbb{R}^N)} \to \infty$ as $n\to\infty$. Hence we have that for every $v \in H^s_\Omega(\mathbb{R}^N)$

$$\frac{C(N,s)}{2}\langle u_n, v\rangle_{H^s_\Omega(\mathbb{R}^N)} = \int_\Omega \left(\frac{\lambda_n f'(\infty)u_n + (1-\lambda_n)f(u_n)}{\|u_n\|}\right) v.$$

Given our assumptions about f we can rewrite it as

$$f(t) = tf'(\infty) + g(t)$$

with $g(t) = o(t)$ as $|t|\to\infty$, getting

$$\frac{C(N,s)}{2}\langle u_n, v\rangle_{H^s_\Omega(\mathbb{R}^N)} = f'(\infty)\int_\Omega \left(\frac{u_n}{\|u_n\|}\right) v + \int_\Omega \left((1-\lambda_n)\frac{g(u_n)}{\|u_n\|}\right) v.$$

Since $H^s_\Omega(\mathbb{R}^N)$ is reflexive we have that $\left\{\frac{u_n}{\|u_n\|}\right\}_{n\in\mathbb{N}}$ converges weakly to some $u \in H^s_\Omega(\mathbb{R}^N)$ up to a subsequence. On the other hand, given $\varepsilon>0$ there exists $K>0$ such that if $|t|>k$ then $g(t)/t < \varepsilon$, thus

$$\left|\int_\Omega \left(\frac{g(u_n)}{\|u_n\|}\right) v\right| \leq \left|\int_{|u_n|\leq K} \left(\frac{g(u_n)}{\|u_n\|}\right) v\right| + \left|\int_{|u_n|>K} \left(\frac{g(u_n)}{u_n}\right) \frac{u_n}{\|u_n\|} v\right|$$
$$C_\Omega \frac{\sup\{|g(t)| \big| t\in[-K,K]\}}{\|u_n\|}\|v\| + \varepsilon\|v\|.$$

Hence, we get

$$\frac{C(N,s)}{2}\langle u_n, v\rangle_{H^s_\Omega(\mathbb{R}^N)} = f'(\infty)\int_\Omega uv, \qquad \forall v \in H^s_\Omega(\mathbb{R}^N).$$

Let us see that $u \neq 0$. Arguing by contradiction let us suppose that $u = 0$, this implies that

$$\frac{C(N,s)}{2}\|u_n\|^2_{H^s_\Omega(\mathbb{R}^N)} = \int_\Omega \left(\lambda_n f'(\infty)u_n + (1-\lambda_n)f(u_n)\right)u_n.$$

Thereby, by the Cauchy-Schwartz inequality we get

$$1 \leq \left(f'(\infty)\left\|\frac{u_n}{\|u_n\|}\right\|_{L^2(\Omega)} + \left\|\frac{(1-\lambda_n)g(u_n)}{\|u_n\|}\right\|_{L^2(\Omega)}\right)\left\|\frac{u_n}{\|u_n\|}\right\|_{L^2(\Omega)}.$$

Notice that our previous estimates show, in fact, that $\left\|\frac{g(u_n)}{\|u_n\|}\right\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. On the other hand, since $\frac{u_n}{\|u_n\|}$ converges strongly to 0 in $L^2(\Omega)$, up to a subsequence, we get

$$\frac{C(N,s)}{2} \leq \left(f'(\infty)\left\|\frac{u_n}{\|u_n\|}\right\|_{L^2(\Omega)} + \left\|\frac{(1-\lambda_n)g(u_n)+1}{\|u_n\|}\right\|_{L^2(\Omega)}\right)\left\|\frac{u_n}{\|u_n\|}\right\|_{L^2(\Omega)} \to 0,$$

as $n \to \infty$. This contradiction shows that $u \neq 0$. This implies that u solves the eigenvalue problem

$$\frac{C(N,s)}{2}\langle u, v\rangle_{H^s_\Omega(\mathbb{R}^N)} = f'(\infty)\int_\Omega uv, \qquad \forall v \in H^s_\Omega(\mathbb{R}^N),$$

which is absurd since $f'(\infty)$ is not an eigenvalue of our problem.

Now we prove that $d(\nabla J, B_R(0), 0) = (-1)^k$. Given the invariance under homotopy of the degree (*Lemma* 107) it suffices to compute the degree of ∇J_1 in $B_R(0)$ with respect to 0. Notice that the only critical point of J_1 is 0, hence, in virtue of *Lemma* 107, it suffices to find the number of negative eigenvalues of $D^2J_1(0)$. Since

$$D^2J_1(0)(u,v) = \mu\langle u, v\rangle_{H^s_\Omega(\mathbb{R}^N)} \implies \frac{C(N,s)}{2} - \mu\langle u, v\rangle = f'(\infty)\int_\Omega uv.$$

It follows that the negative eigenvalues of $D^2J_1(0)$ satisfy

$$\frac{f'(\infty)}{\frac{C(N,s)}{2} - \mu} = \lambda_i(s) \implies \mu = \frac{\frac{C(N,s)}{2}\lambda_i(s) - f'(\infty)}{\lambda_i(s)}.$$

Hence, by our assumption that $\frac{C(N,s)}{2}f'(\infty) \in (\lambda_k(s), \lambda_{k+1}(s))$ it follows that $D^2J_1(0)$ has k negative eigenvalues (counting the algebraic multiplicity), implying that $d(\nabla J, B_R(0), 0) = (-1)^k$. $\square$

Finally, we conclude with the proof of our second theorem.

Proof of Theorem 102. *Lemma* 109 implies that there exist a nontrivial solution $\omega_3 \in C^{0,s}(\mathbb{R}^N)$ such that $d(\nabla J, \omega_3, U) = (-1)^k$ with U an open neighbourhood of ω_3 that does not contain any other critical point. Also notice that, since 0 is a local minimum of J, its local degree is 1 (see [6]). Arguing by contradiction, let us suppose that there are not more solutions. The excision property of the Leray-Schauder degree implies that

$$(-1)^k = d(\nabla J, B_R, 0) = 1 + (-1)^k,$$

which is clearly a contradiction for any value of k. Thus, there exists at least another nontrival critical point of J which we call ω_4. Finally, *Theorem* 99 implies that ω_3 and ω_4 are continuous. □

7. Index of notation

$\mathscr{M}(\Omega;\mathbb{C})$	space of complex-valued measurable functions on $\Omega \subset \mathbb{R}^N$, 13
$\mathscr{M}(\Omega)$	space of real-valued measurable functions on $\Omega \subset \mathbb{R}^N$, 9
$\mathscr{F}$	Fourier transform, 13
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\partial^\alpha, \partial_i, \partial_{ij}$	multi-index notation, 9
$C_B^k(\overline{\Omega}), C_0^k(\Omega), C_0^\infty(\Omega), C_B^\infty(\overline{\Omega})$	spaces of differentiable functions on Ω, 9
$C_B^{k,\gamma}(\overline{\Omega}), C_0^{k,\gamma}(\Omega)$	spaces of Hölder continuous functions on Ω, 9
$\\|\cdot\\|_{C_B^k(\overline{\Omega})}$	norm space of differentiable bounded functions on Ω, 9
$\\|\cdot\\|_{C_B^{k,\gamma}(\overline{\Omega})}$	norm space of Hölder continuous bounded functions on Ω, 9
Q, Q_+, Q_0	local models for smooth domains, 11
$\\|\cdot\\|_{L^p(\Omega)}$	norm space $L^p(\Omega)$
$\\|\cdot\\|_{L^p(\Omega;\mathbb{C})}$	norm space $L^p(\Omega;\mathbb{C})$
$L^p(\Omega)$	space of real-valued p-integrable functions on Ω, 9
$L^p(\Omega;\mathbb{C})$	space of complex-valued p-integrable functions on Ω, 13
$\mathscr{S}(\mathbb{R}^N)$	Schwartz space of rapidly decreasing real-valued functions, 13
$\mathscr{S}(\mathbb{R}^N;\mathbb{C})$	Schwartz space of rapidly decreasing complex-valued functions, 13
$\mathscr{S}'(\mathbb{R}^N)$	topological dual of $\mathscr{S}(\mathbb{R}^N)$
$\mathscr{S}'(\mathbb{R}^N;\mathbb{C})$	topological dual of $\mathscr{S}(\mathbb{R}^N;\mathbb{C})$
Δ	Laplacian operator
$(-\Delta)^s$	fracional Laplacian of order s or s-fractional Laplacian, 20
D^s	homogeneous derivative of order s or s-derivative , 18
δ	second differences operator , 42
$(-\Delta)^s$	fracional Laplacian of order s or s-fractional Laplacian, 20
τ_h	translation operator $(\tau_h f(x) := f(x+h))$, 33
$\lvert\cdot\rvert$	Lebesgue measure

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