Unrecognizability of Manifolds

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1

Introduction

One of the fundamental problems of topology is to decide, given two topological spaces, whether or not they are homeomorphic. This problem is known as the **Homeomorphism Problem.** To effectively answer this question one must first specify how a manifold is described and be sure that such a description is suitable for input into a computing device.

The next step will be to come up with a general effective procedure to answer this question when applied to a sufficiently general class of spaces, of a specific dimension $n \geq 3$ (PL manifolds, smooth manifolds, etc.) In this generality, it turns out that for compact PL manifolds, the homeomorphism problem is undecidable for spaces of dimension $n \geq 4$, as was proved by A.A Markov (Mar58). Furthermore, a dramatic improvement of the previous result was discovered by S. P. Novikov (VKF74, pg.169) in the sense that for $n \geq 5$, it is impossible to recognize the n-sphere, and in fact the same holds for any compact n-dimensional smooth manifold.

The main purpose of this monograph is to present, for those readers with a basic background in algebra and topology, a detailed and accessible proof of S.P. Novikov's result. We follow the exposition that appears in the appendix of (Nab95).

As a guide to the reader we offer an outline of the main points developed in our treatment. First, we prove the algorithmic unrecognizability of the n-sphere for $n \geq 5$, according to the following steps:

1. We start from a finite presentation of a group G with unsolvable word problem.

2. Using the presentation for G we build a sequence of finitely presented groups $\{G_i\}$ such that $\{G_i\}$ is an Adian-Rabin sequence.

3. Following Novikov, we modify the sequence $\{G_i\}$ and obtain a new sequence of finitely presented groups $\{G'_i\}$ which have trivial first and second homology, such that $\{G'_i\}$ is an Adian-Rabin sequence, i.e., we obtain a Novikov sequence.

4. Next, we construct a sequence of compact non-singular algebraic hypersurfaces $S_i \subset \mathbb{R}^{n+1}$, so that S_i is a homology sphere and $\pi_1(S_i) = G'_i$. Moreover this is done in such a way that S_i is diffeomorphic to $\mathbb{S}^n$ if and only if G'_i is trivial. (From The Generalized Poincaré Conjecture and The Characterization of the smooth n-disc $\mathbb{D}^n$, $n \geq 6$.)

5. Finally, arguing by contradiction, we assume that the n-sphere is algorithmically recognizable. Thus, if we apply this presumed algorithm to the elements of the sequence $\{S_i\}$ we could determine which of them are diffeomorphic to the n-sphere. This in turn would allow us to single out the trivial elements of the given Novikov sequence, but this is clearly impossible.

As a final step, we apply the previous result to stablish the unrecognizability of the compact smooth n-manifolds, $n \geq 5$, according to the following steps:

1. Assume for simplicity that M_0 is a connected n-dimensional manifold that can be effectively recognized among the class of all compact n-dimensional manifolds.

2. Fix a compact n-dimensional manifold M effectively generated from a Novikov sequence of groups and define $M_1 = M_0 \# M$.

3. Apply to M_1 the procedure to recognize M_0.

4. If the answer is **No**, then M is not a sphere.

5. If the answer is **Yes**, note that $\pi_1(M) = 1$ and then conclude that M is the sphere, since the only simply connected n-manifold generated from a Novikov sequence is the n-sphere.

6. From 4 and 5, an effective procedure to recognize M_0 will allow us to recognize the n-sphere, which is a contradiction.

2

Geometry and Topology

In this chapter we review some of the fundamental results from geometry and topology that will be needed in the sequel.

2.1 Combinatorial Manifolds

In this section we assume that the reader is familiar with the basic techniques of PL topology. In this regard, the classical reference is of course (RS82). A more concise and modern treatment can be found in (SD01).

Definition. A (possibly infinite) polyhedra $M \subset \mathbb{R}^m$ is said to be a **PL n-manifold**, if each point of M has a closed neighbourhood which is PL homeomorphic with an n-simplex.

Definition. Let X be a topological space. A **triangulation** for X consists of a complex K and a homeomorphism $t : |K| \longrightarrow X$. Two triangulations $t : |K| \longrightarrow X$ and $t' : |K'| \longrightarrow X$ of X are **equivalent** if there is a PL homeomorphism $h : |K| \longrightarrow |K'|$ such that $t' \circ h = t$. When the polyhedron $|K|$ is a PL n-manifold, the tuple $(X, |K|, t)$ is called a **combinatorial n-manifold (or PL n-manifold)**, and the triangulations $t : |K| \longrightarrow X$ is called a **combinatorial structure (or PL structure)** for X.

Now that we have defined the general notion of a combinatorial manifold M, we will describe next what it means for a PL triangulation to be compatible with a differential structure on M.

Definition. Let $t : |K| \longrightarrow M$ be a PL triangulation for $(M, \mathfrak{U})$, with $\mathfrak{U}$ a differential structure on M. We say that the combinatorial structure determined by $(|K|, t)$ is compatible with $\mathfrak{U}$, if for each simplex $\sigma \in K$ there is a chart $\phi : W \longrightarrow \mathbb{H}^n$ in $\mathfrak{U}$ such that $t(\sigma) \subset W$ and $\phi(t(\sigma))$ is a (rectilinear) simplex in $\mathbb{H}^n$.

Remark. It is worthwhile to mention, at this juncture, that, as was prove by S.S Cairns and J.H.C Whitehead, see (Cai35) and (Whi40), every differentiable manifold admits a compatible combinatorial structure.

2.1.1 Regular Neighbourhoods

Given a closed sub-complex C of a combinatorial manifold M, sometimes it is useful to find a neighbourhood of C (in M) which retains as much of the topology of C as possible. What we look for is usually described by topologists as "a small neighbourhood of C" in M. Formally, let $C \subset \text{int}M$ be a closed triangulated subset of a combinatorial manifold M, with $L \subset K$ simplicial complexes such that $\phi : |K| \longrightarrow M$ is a PL triangulation such that $\phi(L) = C$. Let K'' be the second barycentric subdivision of K and $\psi : K \longrightarrow K''$ the corresponding canonical map with $L'' \subset L$ such that $\psi(L'') = L$.

Definition. The **Regular Neighbourhood** N of L is the family of simplices

$$N = \{\sigma \in K'' \,|\, \sigma \cap L'' \neq \varnothing\},$$

in other words $N = \text{Star}(L'', K'')$. The image of $\psi(N)$ under ϕ is a (closed) regular neighbourhood of C and the corresponding interior of N is an open neighbourhood of C.

Remark. A simplex $\sigma \in K''$ is in N if and only if some face of σ is also a simplex in L. This guarantees that the interior of N will be an open set containing L. Finally, by taking the second barycentric subdivision we make sure that the neighbourhood is a "small" set containing L.

For our purposes, the relevant features of the notion of a regular neighbourhood are the following:

(i) If X and M are polyhedra $X \subset M$ with X compact, and M a combinatorial manifold, it is always possible to find a regular neighbourhood of X in M.

(ii) Every regular neighbourhood of X is a compact combinatorial manifold with boundary that retracts onto X.

2.2 Algebraic and Differential Topology

Since our main objective in this monograph is a detailed account of the unsolvability of a problem involving smooth manifolds, we will need to make use of some of the basic tools of algebraic and differential topology. We compile a list of fundamental definitions and results from topology and geometry that will be needed. Good standard references for algebraic and differential topology are (Hat01), (Mau70), (MS74), and (Hir76), where the reader can find proofs of some of the results stated below.

Theorem 2.2.1 (Van Kampen Theorem.) *If X is the union of path-connected open sets A_α each containing the base point $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then the homomorphism: $\phi : *_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ is surjective. In addition if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of ϕ is a normal subgroup N generated by all elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$, and so ϕ induces an isomorphism $\pi_1(X) \approx *_\alpha(A_\alpha)/N$.*

Proposition 2.2.2 *$H^n(K, L) \approx FH_n(K, L) \oplus TH_{n-1}(K, L)$.*

Theorem 2.2.3 (Alexander Duality Theorem.) *If A is a compact subset of $\mathbb{R}^n$, then for all q and R-modules G*

$$\widetilde{H}_q(\mathbb{R}^n \setminus A; G) \approx \widetilde{H}^{n-q-1}(A; G).$$

Theorem 2.2.4 (Poincaré Duality Theorem.) *If X is a closed R-orientable n-manifold, then for all q and R modules G*

$$H_q(X; G) \approx H^{n-q}(X; G).$$

Theorem 2.2.5 *Let X a triangulable path-connected n-manifold. Then $H_n(X, \mathbb{Z}) = \mathbb{Z}$ if X is orientable, and $H_n(X) = 0$ otherwise. In any case, $H_n(X; \mathbb{Z}_2) = \mathbb{Z}_2$.*

Definition. Let X be a path-connected space. Suppose $n \geq 1$ and let i_n be a generator of $H_n(\mathbb{S}^n)$. The **Hurewicz map** $\rho_n : \pi_n(X) \longrightarrow H_n(X)$ is defined by $\rho([f]) := f_*(i_n)$ for a representative $f : S^n \longrightarrow X$.

Theorem 2.2.6 (Hurewicz Isomorphism Theorem.) *If X is $(n-1)$-connected for some $n \geq 2$, then $h_n : \pi_n(X, x_0) \longrightarrow H_n(X,)$ is an isomorphism.*

Theorem 2.2.7 (Whitehead Theorem.) *Let X and Y be path-connected pointed spaces and let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a map. If there is $n \geq 1$ such that*

$$f_\# : \pi_q(X, x_0) \longrightarrow \pi_q(Y, y_0)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$, then

$$f_* : H_q(X, x_0) \longrightarrow H_q(Y, y_0)$$

is an isomorphism for $q < n$. Conversely, if X and Y are simply connected and f_* is an isomorphism for $q < n$ and an epimorphism for $q = n$, then $f_\#$ is an isomorphism for $q < n$ and an epimorphism for $q = n$.

Theorem 2.2.8 (Theorem of Hopf.) *For any path-connected space X with fundamental group G, there is an exact sequence*

$$\pi_2(X) \xrightarrow{\rho} H_2(X; \mathbb{Z}) \longrightarrow H_2(G; \mathbb{Z}) \longrightarrow 0$$

where ρ is the Hurewicz map.

Definition. Let $X \subset \mathbb{R}^n$ be a smooth manifold. For each point $x \in X$ define **the space of normals** to X at x to be

$$N_x(X) = \{v \in \mathbb{R}^n : v \perp TX_x\}.$$

The total normal space $E(\nu_X)$ of X in $\mathbb{R}^n$ is defined by:

$$E(\nu_X) := \{(x, v) \in X \times \mathbb{R}^n : v \perp TX_x\}.$$

Definition. Given a submanifold $X \subset \mathbb{R}^n$ and a continuous function $\epsilon : X \longrightarrow (0, \infty)$, we introduce the following notation,

$$\mathcal{N}_\epsilon(X) = \{(x, v) \in N(X) \,|\, |v| < \epsilon(x)\}.$$

Theorem 2.2.9 (Tubular Neighborhood Theorem.) *Let $X \subset \mathbb{R}^n$ be a proper submanifold without boundary and let U be a neighborhood of X in $\mathbb{R}^n$. Then there exists a continuous function $\epsilon : X \longrightarrow (0, \infty)$ and a diffeomorphism $\phi : \mathcal{N}_\epsilon(X) \longrightarrow V$, onto an open neighborhood V of X in U, such that $\phi(x, 0) = x$ for all $x \in X$. In particular, X is a strong deformation retract of V. Moreover, if X is compact, then we can choose a constant $\epsilon > 0$, such that $V = V_{\epsilon'} = \bigcup_{x \in X} B_{\epsilon'}(x)$.*

Theorem 2.2.10 (Isotopy Extension Theorem.) *Let $f : M \longrightarrow N$ be an imbedding of a manifold M in a manifold N with $\partial N = \varnothing$. Let $K \subset M$ be a compact subset and $F : M \times \mathbb{R} \longrightarrow N \times \mathbb{R}$ be the track of an isotopy of f. Then there is $G : N \times \mathbb{R} \longrightarrow N \times \mathbb{R}$ which is the track of an isotopy such that $G(f(x), t) = F(x, t)$ for all $x \in K$ and $t \in [0, 1]$. Moreover, G is the identity map outside a compact subset of $N \times \mathbb{R}$.*

Definition. (i) A hypersurface imbedded in a manifold is a submanifold of codimension one.

(ii) A homology n-sphere is an n-manifold M with homology groups all isomorphic to those of the n-sphere $\mathbb{S}^n$.

(iii) A homotopy n-sphere is an n-manifold M which it is homotopy equivalent to the n-sphere $\mathbb{S}^n$.

Theorem 2.2.11 (The Jordan-Brouwer Separation Theorem.) *Let X be a compact, connected hypersurface in $\mathbb{R}^{n+1}$. The complement of X in $\mathbb{R}^{n+1}$ consists of two connected open sets, the "outside" D_0 and the "inside" D_1. Moreover, $\overline{D_1}$ is a compact manifold with boundary equal to X.*

Proposition 2.2.12 (Characterization of the smooth n-disc $\mathbb{D}^n$, $n \geq 6$.) *Suppose W^n is a compact simply connected smooth n-manifold, $n \geq 6$, with a simply connected boundary. The following statements are equivalent.*

(i) W^n is diffeomorphic to $\mathbb{D}^n$.

(ii) W^n is homeomorphic to $\mathbb{D}^n$.

(iii) W^n is contractible.

(iv) W^n *has the (integral) homology of a point.*

Proof. See Proposition A in Chapter 9 of (Mil65).

■

Proposition 2.2.13 (The Generalized Poincaré Conjecture.) *If* M *is a closed simply connected smooth* n*-manifold,* $n \geq 5$*, with the (integral) homology of the* n*-sphere* $\mathbb{S}^n$*, then* M *is homeomorphic to* S^n*. If* $n = 5$ *or* 6*,* M *is diffeomorphic to* $\mathbb{S}^n$*.*

Proof. See (Sma61).

■

Theorem 2.2.14 *Let* S *be a smooth homotopy* n*-sphere,* $n \geq 5$*, which is also a hypersurface (i.e., smoothly embeds in* $\mathbb{R}^{n+1}$*). Then* S *is homeomorphic to* $\mathbb{S}^n$ *if and only if* S *is diffeomorphic to* $\mathbb{S}^n$*.*

Proof.

($\Leftarrow$) It is clear.

($\Rightarrow$) If a homotopy sphere is imbedded in Euclidean space as a hypersurface S, then it separates the space into two components. (2.2.11.) The closure W^{n+1} of one of these components is a compact manifold whose boundary is S. (See Theorem 2.2.11.) Additionally, it can be checked that it is contractible. (Using Alexander Duality Theorem 2.2.3, Whitehead Theorem 2.2.7 and Van Kampen's Theorem 2.2.1.). Then, by Proposition 2.2.12 W^{n+1} is diffeomorphic to $\mathbb{D}^{n+1}$ and therefore S is diffeomorphic to $\mathbb{S}^n$.

■

Remark. An application of Whitehead Theorem 2.2.7 and the Hurewicz Isomorphism Theorem 2.2.6 shows that a homology n-sphere, $n > 1$, is a homotopy sphere if and only if it is simply connected.

Corollary 2.2.15 *Let* S *be a smooth homology* n*-sphere,* $n \geq 5$*, which is also a hypersurface. Suppose that* S *has trivial fundamental group. Then* S *is homeomorphic to* $\mathbb{S}^n$ *if and only if* S *is diffeomorphic to* $\mathbb{S}^n$*.*

Corollary 2.2.16 *Let S be a smooth homology n-sphere, $n \geq 5$, which is also a hypersurface. Then S is diffeomorphic to $\mathbb{S}^n$ if and only if $\pi_1(S) = 1$.*

Proof.

($\Rightarrow$) It is clear.

($\Leftarrow$) By Proposition 2.2.13 S is homeomorphic to $\mathbb{S}^n$, and by Corollary 2.2.15, it follows that S is diffeomorphic to $\mathbb{S}^n$.

■

Theorem 2.2.17 (Whitney Embedding Theorems.) *Let N and M be manifolds of dimension n and m respectively, and let $f : N \longrightarrow M$ be a smooth map. Then we have the following:*

(i) If $2n + 1 \leq m$, then f is homotopic to an imbedding $N \hookrightarrow M$, and for $2n + 2 \leq m$, any two homotopic imbeddings are isotopic.

(ii) If $m = 2n \geq 6$ and $\pi_1(M) = \{1\}$, then f is homotopic to an imbedding $N \hookrightarrow M$.

Proof. See (Ran02) Chapter 7.

■

Remark. In this monograph we will make use of the so-called "general position arguments" which form part of the typical reasoning closely associated with the concept of transversality, both in the PL and smooth categories. For more details in the smooth case, we refer the reader to the now classical sources (Bre93), Section 15, Chapter II, and (Hir76), Section 2, Chapter 3. Especially, we will use corollaries 15.6 and 15.7 of the former and theorems 2.4 and 2.5 of the latter. Finally, for the PL category we recommend the standard references (RS82), Chapter 5 and (SD01) Chapter 5.

Finally, as an illustration of a typical argument of general position, we state and prove the following important result. See (Ran02), Lemma 7.28.

Lemma 2.2.18 *Let $f : N \longrightarrow M$ be an immersion with image $V = f(N) \subset M$. If $m \geq 5$ and $m - n \geq 3$ then $\pi_1(M \setminus V) = \pi_1(M)$.*

Proof. The morphism $\pi_1(M \setminus V) \longrightarrow \pi_1(M)$ induced by inclusion is surjective, since every map $\mathbb{S}^1 \longrightarrow M$ can be moved away from V by general position. In order to prove that the morphism is injective consider an element $x \in \ker(\pi_1(M \setminus V) \longrightarrow \pi_1(M))$, which may be represented by a commutative square

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{i} & M \setminus V \\ \downarrow & & \downarrow \\ \mathbb{D}^2 & \xrightarrow{j} & M \end{array}$$

with i an embedding. Since $m \geq 5$ j is homotopic to an embedding (leaving the embedding i fixed). Now ensure that $V \cap j(\mathbb{D}^2) = \varnothing$. By general position move $j(\mathbb{D}^2)$ away from V by an arbitrarily small perturbation leaving j an embedding, and leaving i alone on $\mathbb{S}^1$. The result is an embedded $j(\mathbb{D}^2) \subset M \setminus V$ with $\partial(j(\mathbb{D}^2)) = i(\mathbb{S}^1)$, so that $x = 1 \in \pi_1(M \setminus V)$.

■

2.3 Algebraic and Geometric Surgery

In topology, surgery is a procedure for changing one manifold into another of the same dimension and as suggested by its name, it involves some cutting, removing and replacing. Specifically, suppose we have a smooth n-dimensional manifold M. A surgery on M has the effect of excising a copy of $\mathbb{S}^m \times \mathbb{D}^{n-m}$, $m \leq n$, and replacing it by $\mathbb{D}^{m+1} \times \mathbb{S}^{n-m-1}$, since these two manifolds share the same boundary, $\mathbb{S}^m \times \mathbb{S}^{n-m-1}$.

In order to perform a surgery on a manifold one needs an imbedded product of a sphere, which usually belongs to a specific homology class, and a disk. Having such product is, by virtue of the Tubular Neighborhood Theorem, the same as finding an imbedded sphere with a trivial normal bundle.

We sketch next the basic results connected with bundles and imbeddings that constitute the foundations of surgery theory. We follow the general treatment that appears in (Ran02) and (KM07).

2.3.1 Bundles

Definition. A **Fibre Bundle** is a sequence of spaces and maps

$$F \longrightarrow E \xrightarrow{p} B$$

that is "**locally trivial**" in the following sense: for all $b \in B$ there exists an open neighbourhood $b \in U \subset B$ such that

$$\begin{array}{ccc} \phi : p^{-1}(U) & \longrightarrow & U \times F \\ & \searrow^{p} & \downarrow \mathrm{proy}_1 \\ & & U \end{array}$$

where ϕ is a homeomorphism.
The space B is called the **base space**, E is called the **total space** and the map p is called the **projection** of the bundle. For each $b \in B$, $F = p^{-1}(b)$ is called the **fibre** of the bundle over $b \in B$.

Definition. (i) A k**-plane bundle** or **vector bundle** (X, η) is a fibre bundle

$$\mathbb{R}^k \longrightarrow E(\eta) \xrightarrow{p} X$$

such that

(a) each fibre $\eta(x) = p^{-1}(x)$, $x \in X$, is a k-dimensional real vector space,

(b) for each $x \in X$ the homeomorphism $\phi : U \times \mathbb{R}^k \longrightarrow p^{-1}(U)$ is such that for each $u \in U$ the restriction of ϕ to $\{u\} \times \mathbb{R}^k$ is a isomorphism of vector spaces.

(ii) A **bundle map** $(f, b) : (X', \eta') \longrightarrow (X, \eta)$ is a commutative diagram of maps

$$\begin{array}{ccc} E(\eta') & \xrightarrow{b} & E(\eta) \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

such that

$$\begin{array}{rrcl} b(x') := b|\eta'(x') : & \eta'(x') & \longrightarrow & \eta(f(x')) \\ & v & \mapsto & b(v) \end{array}$$

is a linear map of vector spaces for each $x' \in X'$.

Definition. (i) The **pullback bundle** or **induced bundle** of a k-plane bundle η over X along a map $f : X' \longrightarrow X$ is the k-plane bundle $f^*\eta$ over X' defined by

$$E(f^*\eta) = \{(x', y) \in X' \times E(\eta) \,|\, f(x') = p(y) \in X\};$$

with projection map

$$\begin{array}{rccc} p' : & E(f^*\eta) & \longrightarrow & X' \\ & (x', y) & \mapsto & x', \end{array}$$

and fibres

$$f^*\eta(x') = \eta(f(x')),\ x' \in X'.$$

(ii) A **pullback bundle map** is a bundle map $(f, b) : (X', \eta') \longrightarrow (X, \eta)$ such that each of the linear maps

$$\begin{array}{rrcl} b(x') := b|\eta'(x') : & \eta'(x') & \longrightarrow & \eta(f(x')) \\ & v & \mapsto & b(v) \end{array}$$

is an isomorphism of vector spaces, i.e, such that the function

$$\begin{array}{ccc} E(\eta') & \longrightarrow & E(f^*\eta) \\ y & \mapsto & (p'(y), b(y)) \end{array}$$

is a homeomorphism.

(iii) An **isomorphism** $b : \eta' \longrightarrow \eta$ of bundles over the same space X is a pullback bundle map of the type $(1, b) : (X, \eta') \longrightarrow (X, \eta)$.

Definition. (i) A k-plane bundle η over a space X is **trivial** if it is isomorphic to the trivial bundle ϵ^k with projection

$$\begin{array}{rccc} p : & E(\epsilon^k) = X \times \mathbb{R}^k & \longrightarrow & X \\ & (x, y) & \mapsto & x. \end{array}$$

(ii) A **framing** (or **trivialisation**) of a k-plane bundle η is an isomorphism to the k-plane bundle ϵ^k.

(iii) The **Whitney sum** of a j-plane bundle α and a k-plane bundle β over X is the $(j+k)$-plane bundle $\alpha \oplus \beta$ over X defined by

$$E(\alpha \oplus \beta) = \{(u, v) \in E(\alpha) \times E(\beta) \,|\, p_\alpha(u) = p_\beta(v) \in X\}$$

with fibres

$$(\alpha \oplus \beta)(x) = \alpha(x) \oplus \beta(x),\ x \in X.$$

Definition. Let η be a k-plane bundle over a space X. The **disk bundle** of η is

$$D(\eta) = \{v \in E(\eta) \,|\, \|v\| \leq 1\}.$$

Definition. Let $1 \leq k \leq n$.

(i) The **Grassmann manifold** $G_k(\mathbb{R}^n)$ is the set of all k-dimensional subspaces of $\mathbb{R}^n$.

(ii) The **canonical k-plane bundle** over $G_k(\mathbb{R}^n)$, given by

$$\gamma_k(\mathbb{R}^n) = \{(W, x) \,|\, W \in G_k(\mathbb{R}^n), x \in W\},$$

has projection

$$\begin{array}{ccc} \gamma_k(\mathbb{R}^n) & \longrightarrow & G_k(\mathbb{R}^n) \\ (W, x) & \mapsto & W. \end{array}$$

Now the inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \subset \dots$ induce, in turn, inclusions of the corresponding Grassmann manifolds, which allows us to take the direct limit

$$BO(k) := \varinjlim G_k(\mathbb{R}^n) = G_k(\mathbb{R}^\infty).$$

Now we have set the stage for one of the main results in the theory of vector bundles.

Theorem 2.3.1 (Bundle Classification Theorem.) *Let X be a finite CW complex.*

(*i*) *Every k-plane bundle over X is isomorphic to the pullback $f^*\gamma_k(\mathbb{R}^\infty)$ of the canonical k-plane bundle $1_k = \gamma_k(\mathbb{R}^\infty)$ over the classifying space $BO(k) = G_k(\mathbb{R}^\infty)$ along a map $f : X \longrightarrow BO(k)$:*

$$\begin{array}{ccc} f^*\gamma_k(\mathbb{R}^\infty) & \longrightarrow & \gamma_k(\mathbb{R}^\infty) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BO(k). \end{array}$$

(*ii*) *There are bijections:*

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ k\text{-plane bundles over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Homotopy classes of maps} \\ X \longrightarrow BO(k). \end{array} \right\}$$

(*iii*) *The trivial k-plane bundle ϵ^k is classified by the trivial map $\{*\} : X \longrightarrow BO(k)$.*

(*iv*) *The pullback $f^*\eta$ of a k-plane bundle $\eta : X \longrightarrow BO(k)$, along a map $f : X' \longrightarrow X$, is classified by the composite $f^*\eta : X' \xrightarrow{f} X \xrightarrow{\eta} BO(k)$.*

Proof. See Chapter 5 of (MS74).

■

Remark. The above classification theorem allows to talk indistinctly of vector bundles and their classifying maps.

Proposition 2.3.2 (*i*) *Every vector bundle $\eta : X \longrightarrow BO(k)$ has a stable inverse, i.e., a vector bundle $-\eta : X \longrightarrow BO(j)$, j large, such that*

$$\eta \oplus -\eta = \epsilon^{j+k} : X \longrightarrow BO(j+k).$$

(*ii*) *A k-plane bundle η can be framed, is trivial, if and only if the classifying map $\eta : X \longrightarrow BO(k)$ is null-homotopic.*

Definition. (i) A **stable isomorphism** between a k-plane bundle η and a l-plane bundle η' over the same space X is a bundle isomorphism

$$b : \eta \oplus \epsilon^j = \eta' \oplus \epsilon^r$$

for some $j, r \geq 0$ with $j + k = r + l$.

(ii) A **stable bundle** over X is an equivalence class of bundles η over X, subject to the equivalence relation

$$\eta \sim \eta' \text{ if there is stable isomorphism } \eta \oplus \epsilon^j = \eta' \oplus \epsilon^r \text{ for some } j, r \geq 0.$$

(iii) A k-plane bundle η is **stably trivial** if $\eta \oplus \epsilon^j$ is trivial for some $j \geq 0$.

(iv) X is said **stably parallelizable**, abbreviated s-parallelizable, if its tangent bundle is stably trivial.

Lemma 2.3.3 *Let ξ^k be a k-dimensional vector bundle over a complex K of dimension $p < k$. If the Whitney sum of ξ^k with a trivial bundle ϵ^r is trivial, then ξ^k itself is trivial.*

Proof. See Lemma 4 of (Mil61).

■

Given two vector bundles $\xi : X \longrightarrow BO(j)$, $\zeta : X \longrightarrow BO(k)$ we can form a $(j+k)$-plane bundle by taking the Whitney sum $\xi \oplus \zeta$. This behaves nicely since:

$$\begin{array}{ccc} BO(j) \oplus BO(k) & \hookrightarrow & BO(j+k) \\ (\xi, \zeta) & \mapsto & x \end{array}$$

Thus $\xi \oplus \zeta : X \longrightarrow BO(j) \oplus BO(k) \subset BO(j+k)$. This suggests that we can form the direct limit with respect to the inclusions $BO(k) \longrightarrow BO(k+1)$ by passing from ξ to (ξ, ζ). Hence

$$BO := \varinjlim BO(k).$$

Proposition 2.3.4 *Let X be a finite CW complex and $BO = \varinjlim BO(k)$ the classifying space. Then there is a bijection between the isomorphism classes of stable vector bundles over X and the homotopy classes of maps $X \longrightarrow BO$.*

Proof. See Proposition 5.31 of (Ran02).

■

Proposition 2.3.5 (i) *The pair $((BO(k+1), BO(k)))$ is k-connected.*

(ii) *If $k > m$ then two k-plane vector bundles η, η' over an m-dimensional finite CW complex X are isomorphic if and only if they are stably isomorphic.*

Proof. See Proposition 5.33 of (Ran02).

■

2.3.2 The tangent and normal bundles

Definition. (i) Let M be an m-dimensional manifold with atlas $\mathfrak{U}$. We define the **tangent bundle** of M as the m-plane bundle $\tau_M : M \longrightarrow BO(m)$ with total space the open $2m$-dimensional manifold

$$E(\tau_M) = \left(\coprod_{(U,\phi)\in\mathfrak{U}} U \times \mathbb{R}^m \right) / \sim$$

where

$$(x \in U, h \in \mathbb{R}^m) \sim (x' \in U', h' \in \mathbb{R}^m)$$

if

$$x = x' \in U \cup U' \subset M,\; d(\phi'^{-1}\phi)_{\phi^{-1}}(h) = h',$$

and projection map

$$\begin{array}{rccc} p: & E(\tau_M) & \longrightarrow & M \\ & (x,h) & \mapsto & x \end{array}$$

The **tangent space** to $x \in M$ is the m-dimensional vector space

$$\tau_M(x) = \left(\coprod_{(U,\phi)\in\mathfrak{U},\, x\in U} \{x\} \times \mathbb{R}^m \right) / \sim$$

such that

$$E(\tau_M) = \bigcup_{x\in M} \tau_M(x).$$

(ii) The **differential** of a given differentiable map $f : N \longrightarrow M$ is the bundle map $df : \tau_N \longrightarrow \tau_M$ given by

$$(x \in V, h \in \mathbb{R}^n) \mapsto (f(x) \in U, d(\phi^{-1} f \psi)_{\psi^{-1}(x)}(h)).$$

Remark. An immersion $f : N \longrightarrow M$ induces an injection of tangent spaces $df_x : \tau_N(x) \longrightarrow \tau_M(f(x))$. Therefore it is possible to identify $\tau_N(x)$ with a subspace of $\tau_M(f(x))$. Choosing a metric on M we can define an inner product

$$\begin{array}{rccc} \langle\, , \,\rangle : & \tau_M(f(x)) \times \tau_M(f(x)) & \longrightarrow & \mathbb{R} \\ & (v,w) & \mapsto & \langle v,w\rangle \end{array}$$

such that the orthogonal complement of $\tau_N(x)$

$$\tau_N(x)^{\perp} = \{v \in \tau_M(f(x)) \,|\, \langle v, \tau_N(x) \rangle = 0\},$$

is a subspace and thus, there is a corresponding Whitney sum decomposition

$$\tau_N(x) \oplus \tau_N(x)^{\perp} = \tau_M(f(x)).$$

Definition. The normal bundle $\nu_f : N \longrightarrow BO(m-n)$ of an immersion $f : N \longrightarrow M$ is the $(m-n)$-plane bundle over N with total space

$$E(\nu_f) = \bigcup_{x \in N} \nu_{f(x)}$$

where

$$\nu_{f(x)} = \tau_N(x)^{\perp} \subset \tau_M(f(x)),$$

and is such that there is a Whitney sum decomposition

$$\tau_N(x) \oplus \nu_f = f^* \tau_M = N \longrightarrow BO(m).$$

Definition. A **framing** of an immersion $f : N \longrightarrow M$ is a framing of the normal bundle $\nu_f : N \longrightarrow BO(m-n)$

$$b : \nu_f \simeq \epsilon^{m-n}.$$

Next we want to discuss normal bundles independently of immersions.

Definition. Let M be a m-dimensional manifold.

(i) Let $f : M \longrightarrow \mathbb{S}^{m+k}$, $k \geq 1$, be an imbedding satisfying that $\tau_M \oplus \nu_f = \epsilon^{m+k} : M \longrightarrow BO(m+k)$. We define a **normal bundle** of M by $\nu_M := \nu_f : M \longrightarrow BO(k)$.

(ii) The **stable normal bundle** of M is the unique map $\nu_M : M \longrightarrow BO$ represented by the normal k-bundle $\nu_M : M \longrightarrow BO(k)$ of any imbedding $M \longrightarrow \mathbb{S}^{m+k}$, k large, such that

$$\tau_M \oplus \nu_M = \epsilon^{\infty} : M \longrightarrow BO.$$

Proposition 2.3.6 *Every compact hypersurface M is s-parallelizable.*

Proof. By the Jordan-Brouwer Separation Theorem 2.2.11, M bounds a compact submanifold of dimension $n+1$ in $\mathbb{R}^{n+1}$. Therefore, M is orientable. Now, consider the **Gauss mapping**

$$g : M \longrightarrow \mathbb{S}^n$$

which assigns to each $p \in M$ the outward unit normal vector at x, i.e., $g(x)$ is the unit length, outward pointing vector in $TM_x^{\perp}$. Now define

$$\begin{array}{rccl} h: & M \times \mathbb{R} & \longrightarrow & E(\nu_M) \\ & (x,t) & \mapsto & h(x,t) = (x, tg(x)). \end{array}$$

The function h is smooth, bijective, and has smooth inverse given by $h^{-1}(x,v) = (x, v \cdot g(x))$. Thus $E(\nu_M)$ is diffeomorphic to ϵ^1. Therefore $\tau_M \oplus \epsilon^1 = \tau_{\mathbb{R}^{n+1}} = \epsilon^{n+1}$.

■

Example. The tangent bundle of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. The total space of this bundle is the set

$$E = \left\{(x,v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \,|\, x \perp v\right\}$$

and the corresponding map is $p : E \longrightarrow \mathbb{S}^n$ defined by $p(x,v) = x$. To construct local trivializations, for each $x \in \mathbb{S}^n$, let $U_x \subset \mathbb{S}^n$ be the open hemisphere containing x and bounded by the hyperplane through the origin orthogonal to x. Define $h_x : p^{-1}(U_x) \longrightarrow U_x \times p^{-1}(x)$ by $h_x(y,v) = (y, \pi_x(v))$, with π_x the orthogonal projection onto the hyperplane $p^{-1}(x)$. Then h_x is a local trivialization, since π_x restricts to an isomorphism of $p^{-1}(y)$ onto $p^{-1}(x)$, for each $y \in U_x$.

Example. The normal bundle of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. This bundle has total space

$$E = \left\{(x,v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \,|\, v \perp T\mathbb{S}^n_x \Longleftrightarrow v = tx, \text{ for some } t \in \mathbb{R}\right\}$$

and map $p : E \longrightarrow \mathbb{S}^n$ defined by $p(x,v) = x$. Note that in order to construct local trivializations the functions $h_x : p^{-1}(U_x) \longrightarrow U_x \times \mathbb{R}$ can be obtained by orthogonal projection of the fibers $p^{-1}(y)$ onto $p^{-1}(x)$, for $y \in U_x$.

Proposition 2.3.7 *The normal bundle $\nu_f : \mathbb{S}^n \longrightarrow BO(m-n)$ of an immersion $g : \mathbb{S}^n \longrightarrow M$ is such that:*

$$\nu_f \oplus \tau_{\mathbb{S}^n} = f^* \tau_M \in \pi_n(BO(m)),$$

$$\nu_f \oplus \epsilon^{n+1} = f^*(\tau_M \oplus \epsilon) \in \pi_n(BO(m+1)),$$
$$\nu_f = -f^*(\nu_M) \in \pi_n(BO).$$

Theorem 2.3.8 (Tubular Neighbourhood Theorem.) *Let N and M be manifolds of dimension n and m respectively. An imbedding (immersion) $f : N \longrightarrow M$ extends to a codimension 0 imbedding (immersion) $E(\nu_f) \longrightarrow M$ of the total space of the $(m-n)$-plane bundle $\nu_f : N \longrightarrow BO(m-n)$.*

Proof. See Theorem 11.1 (MS74).

■

Remark. In particular, if ν_f is trivial we can imbed (immerse) the disk bundle $D(\nu_f) \simeq N \times \mathbb{D}^{m-n} \longrightarrow M$.

Proposition 2.3.9 *There is a bijection between the framings (if any) of an imbedding $f : N \longrightarrow M$ and the extensions of f to an imbedding $\overline{f} : N \times \mathbb{D}^{m-n} \longrightarrow M$.*

Proof. See Proposition 5.58 of (Ran02).

■

2.3.3 Surgery

With the previous array of results at our disposal, we set the machinery of surgery in motion.

Definition. An $(m+1)$-dimensional cobordism $(W; M, M')$ is an $(m+1)$-dimensional manifold W with boundary the disjoint union of closed m-dimensional manifolds M, M'.

Definition. Let N and M be smooth manifolds of dimension n and m respectively and $f : N \longrightarrow M$ an imbedding. Then,

(i) f is a **framed** imbedding if it extends to an imbedding

$$\overline{f} : N \times \mathbb{D}^{m-n} \longrightarrow M.$$

(ii) The imbedding

$$f = \overline{f}|N \times \{0\} : N \times \{0\} \longrightarrow M$$

is called the **core** of the framed imbedding.

Definition. Given an $(m+1)$-dimensional manifold with boundary $(W, \partial W)$ and a framed imbedding $\mathbb{S}^{i-1} \times \mathbb{D}^{m-(i-1)} \longrightarrow \partial W$, $0 \leq i \leq m+1$, we define the $(m+1)$-dimensional manifold with boundary $(W', \partial W')$, obtained from W by **attaching an i-handle**, to be the space

$$W' = W \bigcup_{\mathbb{S}^{i-1} \times \mathbb{D}^{m-i+1}} \mathbb{D}^i \times \mathbb{D}^{m-i+1}.$$

Definition. An n-**surgery** on an m-dimensional manifold M removes the image of a framed n-imbedding $\overline{f} : \mathbb{S}^n \times \mathbb{D}^{m-n} \longrightarrow M$ and replaces it with $\mathbb{D}^{n+1} \times \mathbb{S}^{m-n-1}$. The corresponding **effect** of this surgery is the m-dimensional manifold

$$M' = \overline{(M \setminus \overline{g}(\mathbb{S}^n \times \mathbb{D}^{m-n}))} \bigcup_{\mathbb{S}^n \times \mathbb{S}^{m-n-1}} \mathbb{D}^{n+1} \times \mathbb{S}^{m-n-1}.$$

Even more, the n-surgery **kills** the corresponding homotopy class $[f] \in \pi_n(M)$ of the core.

Definition. The **trace** of the n-surgery on $\mathbb{S}^n \times \mathbb{D}^{m-n} \subset M$ is the $(m+1)$-dimensional cobordism $(W; M, M')$ obtained by attaching an $(n+1)$-handle $\mathbb{D}^{n+1} \times \mathbb{D}^{m-n}$ to $M \times I$ at $\mathbb{S}^n \times \mathbb{D}^{m-n} \times \{1\}$.

Theorem 2.3.10 *Let M be an m-dimensional manifold. The following conditions on an element $x \in \pi_n(M)$ are equivalent:*

(i) x can be killed by an n-surgery on M,

(ii) x can be represented by a framed n-imbedding $\overline{f} : N \times \mathbb{D}^{m-n} \longrightarrow M$,

(iii) x can be represented by an n-imbedding $f : \mathbb{S}^n \longrightarrow M$ with trivial normal bundle $\nu_f : \mathbb{S}^n \longrightarrow BO(m-n)$.

Proof. *(i)* $\Longleftrightarrow$ *(ii)* it follows easily from the definitions involve.
(ii) $\Longleftrightarrow$ *(iii)* by Proposition 2.3.9 there is a a bijection between the framings (if any) of an imbedding $f : \mathbb{S}^n \longrightarrow M$ and extensions of f to an imbedding $\overline{f} : \mathbb{S}^n \times \mathbb{D}^{m-n} \longrightarrow M$.

■

Next we observe that, below the middle dimension, the possibility of killing an element of $\pi_n(M)$ is completely determined by the stable normal bundle $\nu_M : M \longrightarrow BO$:

Corollary 2.3.11 *If $2n < m$ an element $x \in \pi_n(M)$ can be killed by a surgery if and only if $(\nu_M)_* x$[1] $= 0 \in \pi_n(BO)$.*

Proof. By the Whitney Embedding Theorems 2.2.17 there is an imbedding $f : \mathbb{S}^n \longrightarrow M$. Now, stabilise the identity $\nu_f \oplus \tau_{\mathbb{S}^n} = f^* \tau_M$ by adding ϵ

$$\nu_f \oplus (\tau_{\mathbb{S}^n} \oplus \epsilon) = f^*(\tau_M \oplus \epsilon)$$

and use that the sphere is s-parallelizable, $\tau_{\mathbb{S}^n} \oplus \epsilon = \epsilon^{n+1}$, in the last equality

$$\nu_f \oplus \epsilon^{n+1} = f^*(\tau_M \oplus \epsilon).$$

We stabilise further by adding $f^*\nu_M$ and obtain a stable isomorphism

$$(\nu_f \oplus f^*\nu_M) \oplus \epsilon^{n+1} = f^*(\tau_M \oplus \epsilon) \oplus f^*\nu_M = f^*((\tau_M \oplus \nu_M) \oplus \epsilon) = \epsilon^{\infty},$$

hence $\nu_f \oplus f^*\nu_M$ is stably trivial.

It follows that the vanishing of $f^*\nu_M = (\nu_M)_* x \in \pi_n(BO)$ is equivalent to the vanishing of $\nu_f \in \pi_n(BO)$. But this is equivalent to $\nu_f = 0 \in \pi_n(BO(m-n))$ as $\pi_n(BO(m-n)) = \pi_n(BO)$.

■

Let M be an m-dimensional manifold and $S \subset M$ a $(k-1)$-dimensional sphere, imbedded with a trivial normal bundle in the interior of M . The following proposition gives conditions guaranteeing that the subgroup of $H_{k-1}(M)$, generated by the corresponding homology class of S, can be killed by a surgery.

Proposition 2.3.12 *If $i = k-1$ and $m \geq 2k$,*

$$H_i(M') = H_i(M)/\left[S\right].$$

[1] Here if x is represented by $g : \mathbb{S}^n \longrightarrow M$, then $f^*\nu_M : \mathbb{S}^n \xrightarrow{g} M \xrightarrow{\nu_M} BO$ is what we denoted by $(\nu_M)_* x$.

Proof. See Proposition 1.1 in Chapter X of (KM07).

■

Remark. Surgery is usually described as a process that generates topological manifolds but, in fact, the process can be performed carefully enough so that the resulting manifolds will be smooth. Topologists do not usually worry about smoothing a topological manifold, resulting from a surgery, due to the fact that such manifolds can always be endowed with a unique compatible differential structure. For details, we recommend (Hir76), Chapter 8, Section 2.

2.3.4 The Whitney Trick

Suppose that we have an immersion, for example, of the circle in the plane and there are "places" where the image crosses itself. In certain situations it is desirable to get rid of these crossings by simply "sliding" the function so that it becomes a smooth imbedding (we should think in terms of isotopies). It is not difficult to image the analogous situation in the n-dimensional case and, in general, it turns out that getting rid of these crossings is not an easy task. In an attempt to solve this problem H. Whitney came up with a famous method, now known as the **Whitney trick**, in his proof of his fundamental embedding theorems. The trick amounts to taking these self-intersection points in pairs and sliding the manifold through itself in order to eliminate them two at a time. (If there is an odd number to begin with, then it must first introduce an extra one.)

A detailed treatment of the conditions justifying the application of Whitney's move, would take us to far afield. An excellent treatment appears, for example, in (Ran02) Chapter 7, especially the first three sections, where the interested reader can find all the relevant details.

2.4 An important notion from Riemannian Geometry

Finally we introduce a basic notion from Riemannian Geometry that will be use of the proof of our main result. For a detailed treatment see (Pet06).

Definition. Let M be a smooth manifold. By a vector field on M we mean a smooth map $\sigma : M \longrightarrow E(\tau_M)$ such that $p \circ \sigma = \mathrm{Id}_M$, where p is the projection of the tangent bundle of M.

Definition. A **Riemannian Manifold** is a smooth manifold M together with a choice of an inner product $\langle\, , \rangle_p$ in each TM_p such that, if X and Y are two smooth vector fields then the function

$$\begin{array}{rccl} g(X,Y): & M & \longrightarrow & \mathbb{R} \\ & p & \mapsto & g(X,Y)_p = \langle X(p), Y(p)\rangle_p \end{array}$$

is smooth. We shall denote this Riemannian manifold with the pair (M, g). The function g is called a Riemannian metric.

Remark. The smooth manifold $\mathbb{R}^n$, with the standard inner product at each tangent space, is a Riemannian manifold.

Definition. Let $f : M \longrightarrow N$ be an immersion and $(N, \overline{g})$ a Riemannian manifold. Then $g(X,Y) := \overline{g}(df_p(X), df_p(Y))_{f(p)}$ defines the **induced (pullback)** Riemannian metric on M.

Definition. Definition Let $\gamma : (a, b) \longrightarrow M$ be a curve, then the length of γ is defined to be

$$l(\gamma) := \int_a^b ||\gamma'(t)||dt,$$

where $|| \cdot ||$ is the induced norm of the inner product.

Theorem 2.4.1 *Let (M, g) be a connected Riemannian manifold. Let $x, y \in M$. Then*

$$d_M(x, y) := \inf\{l(\gamma)\,|\,\gamma \textit{ is a curve joining } x \textit{ to } y\}$$

defines a metric on M.

Definition. Definition Let M be a Riemannian manifold. A curve $\gamma : (a, b) \longrightarrow M$ is a geodesic if:

(i) γ has constant speed.

(ii) for all $t \in (a, b)$ there exists an $\epsilon > 0$ such that $d_M(\gamma(x), \gamma(y)) = l(\gamma|[x, y])$, for all $x, y \in (t - \epsilon, t + \epsilon)$.

Theorem 2.4.2 *Let M be a Riemannian manifold. The for any $x \in M$ and any $v \in TM_x$, there is a geodesic $\gamma_{xv} : (-\epsilon, \epsilon) \longrightarrow M$ such that $\gamma_{xv}(0) = x$ and $\gamma'_{xv}(0) = v$.*

Based on the last theorem, we can finally define a very important map from the tangent bundle of a manifold M to itself:

We let

$$\mathcal{E} := \{(x, v) \in TM \,|\, \gamma_{xv} \text{ is defined on an interval containing } [0, 1]\}$$

be the domain of the **exponential map**, defined by

$$\begin{array}{rccl} \exp : & \mathcal{E} & \longrightarrow & M \\ & (x, v) & \mapsto & \exp(x, v) = \gamma_{xv}(1). \end{array}$$

Definition. For each $x \in M$ we define the **injectivity radius of** (M, g) **at** x to be

$$\text{inj}_x(M, g) = \sup\{r \,|\, \gamma_{xv} \text{ is injective on } B_r(0) \subset TM_x\}$$

and the **injectivity radius of** (M, g) to be

$$\text{inj}(M, g) = \inf\{\text{inj}_x(M, g) \,|\, x \in M\}.$$

Definition. Let N be a submanifold of a Riemannian manifold M and consider the tangent bundle τ_N, which is a sub-bundle of the restriction $\tau_M|N$, then the orthogonal complement $\tau_N^\perp \subset \tau_M|N$ is called the **normal bundle** ν of N in M.

Proposition 2.4.3 *For any smooth submanifold N of a smooth Riemannian manifold M the normal bundle ν is defined, and*

$$\tau_N \oplus \nu = \tau_M|N.$$

Definition. Let N be a submanifold of a Riemannian manifold M. We define the **normal exponential map** for M as the exponential map restricted to the normal bundle ν of N in M.

Remark. Informally, with M is compact, we can think of the injectivity radius of the normal exponential map for M as the maximal radius of the nonselfintersecting open tubular neighbourhood around M.

3

Algebra

In this chapter we introduce the machinery necessary to state and prove some fundamental results from group theory and K-theory that will be needed in the next chapters. For some of the details, we refer the reader to the standard references (Ros78) and (Ros94).

We start with some basic results describing well known fundamental properties of the commutator subgroup:

Theorem 3.0.4 *Let $K \leq H \leq G$ with $K \trianglelefteq G$. Then*

(i) $H \trianglelefteq G$ if and only if $[H, G] \leq H$.

(ii) $H/K \leq Z(G/K)$ if and only if $[H, G] \leq K$.

Proof. See (Ros78), exercise 162.

∎

Theorem 3.0.5 *Let $K \trianglelefteq G$.*

(i) If $x, y \in G$ then, in G/K,

$$[xK, yK] = [x, y]K.$$

(ii) If $H, J \leq G$

$$[HK/K, JK/K] = [H, J]\, K/K.$$

In particular, $[G/K, G/K] = [G, G]K/K$.

Proof. See (Ros78), exercise 164.

■

The next two statements capture some basic properties of free groups. For the corresponding proofs we recommend, for example, (Rot94), Chapter 11.

Theorem 3.0.6 *Let F be the free (not necessarily abelian) group on a set X and $[F,F]$ the commutator subgroup of F. Then*

(*i*) $[F,F] \trianglelefteq F$ *and* $F/[F,F]$ *is abelian.*

(*ii*) $F/[F,F]$ *is free abelian of rank* $|X|$.

Theorem 3.0.7 (The Projective Property of Free Groups.) *Let F be a free group and let G and H be two groups. Assume that $\alpha : F \longrightarrow G$ is a homomorphism and $\beta : H \longrightarrow G$ an emorphism. Then there is a homomorphism $\gamma : F \longrightarrow H$ such that $\alpha = \beta\gamma$.*

Definition. Let G be a group and let A be an abelian group. A **central extension** of G by A is a pair (E, ϕ) such that

(1) E is group,

(2) ϕ is a homomorphism $E \longrightarrow G$ such that

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

is exact,

(3) $A \subset Z(E) = \{e \in E |\, ex = xe \,\text{for all}\, x \in E\}$.

Remark. The class of central extensions of G generates a category, which we can formalize as follows:

(1) (E, ϕ) is an object of the category,

(2) For (E, ϕ) and (E', ϕ'), a morphism $(E, \phi) \longrightarrow (E', \phi')$ is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & G \\ {\scriptstyle\psi}\downarrow & & \| \\ E & \xrightarrow{\phi'} & G. \end{array}$$

Definition. Let $\mathcal{C}$ be the category mentioned in the previous remark.

(1) A central extension (E, ϕ) of G by A is called trivial if it is isomorphic in $\mathcal{C}$ to $G \times A \xrightarrow{p_1} G$.

(2) A central extension (E, ϕ) of G is universal if for any other central extension (E', ϕ') of G, there is a **unique** morphism $(E, \phi) \longrightarrow (E', \phi')$.

Now we can state and prove the main result of this section. See (Ros94), Theorem 4.1.3

Theorem 3.0.8 *Let G be a group. Then we have the following:*

(1) *G has a universal central extension if and only if it is perfect, that is, $G = [G, G]$.*

(2) *Assuming that G is perfect, a central extension (E, ϕ) of G is universal if and only if the following two conditions hold:*

  (*i*) *E is perfect.*

  (*ii*) *all central extensions of E are trivial.*

  Even more, if (i) and (ii) hold and

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

  is a presentation of G then, the universal central extension (E, ϕ) can be constructed as $E = [F, F]/[F, R]$, with

$$\phi : [F, F]/[F, R] \longrightarrow [F, F]/R = [F/R, F/R] = [G, G] = G$$

  the quotient map.

Proof. (1) ($\Rightarrow$):

If G is not perfect, $G/[G, G] \neq 0$. Let $\psi : G \longrightarrow G/[G, G]$ be the quotient map. Now if (E, ϕ) is a central extension of G, we can construct two distinct morphisms from (E, ϕ) to the trivial extension $(G \times G/[G, G], p_1)$,

$$\begin{array}{ccc} E & \xrightarrow{\phi} & G \\ {\scriptstyle \delta=(\phi,1)}\downarrow & & \| \\ G\times G/[G,G] & \xrightarrow{p_1} & G \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\phi} & G \\ {\scriptstyle \delta'=(\phi,\psi\circ\phi)}\downarrow & & \| \\ G\times G/[G,G] & \xrightarrow{p_1} & G. \end{array}$$

This shows that (E, ϕ) cannot be universal. Hence, for G to have a universal central extension, G must be perfect.

(2) ($\Leftarrow$):

Suppose G is perfect. Let (E, ϕ) be a central extension of G satisfying (i) and (ii) and let (E', ϕ') be an arbitrary central extension of G.
(Uniqueness of a morphism.) Suppose $\psi, \psi' : (E, \phi) \longrightarrow (E', \phi')$ are two morphisms of central extensions

$$\begin{array}{ccc} E & \xrightarrow{\phi} & G \\ {\scriptstyle \psi}\downarrow{\scriptstyle \psi'} & & \| \\ E' & \xrightarrow{\phi'} & G. \end{array}$$

Let $x \in E$. Then $(\phi' \circ \psi)(x) = (\phi' \circ \psi')(x)$ so that $\psi(x) = c_x\psi'(x)$ for some $c_x \in A' = \ker(\phi')$. Similarly, if $y \in E$, then $\psi(y) = c_y\psi'(y)$ for some $c_y \in A'$. Hence,

$$\begin{aligned} \psi([x, y]) &= [\psi(x), \psi(y)] \\ &= [c_x\psi'(x), c_y\psi'(y)] \\ &= c_x\psi'(x)c_y\psi'(y)(c_x\psi'(x))^{-1}(c_y\psi'(y))^{-1} \\ &= c_xc_x^{-}1c_yc_y^{-}1\psi'(x)\psi'(y)(\psi'(x))^{-1}(\psi'(y))^{-1} \\ &= [\psi'(x), \psi'(y)] \\ &= \psi'([x, y]) \end{aligned}$$

because c_x and c_y lie in the center. Hence ψ and ψ' coincide on commutators. Since $E = E'$ by (i), ψ and ψ' coincide on all of E.
(Existence of a morphism.) We construct a morphism $\psi : (E, \phi) \longrightarrow (E', \phi')$. Consider

$$E'' = E \times_G E' = \{(x, y) \in E \times E' \mid \phi(x) = \phi(y)\}.$$

Since ϕ and ϕ' are surjective, the projection p_1 on the first factor is a surjective homomorphism from E'' to E. Thus we have a commutative diagram

$$\begin{array}{ccc} E'' & \xrightarrow{p_1} & E \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow \\ E' & \xrightarrow[\phi\circ p_1=\phi'\circ p_2]{\phi'} & G. \end{array} \tag{3.1}$$

Note that since

$$\ker(p_1) = \{(1, y) \in E \times E' \mid y \in \ker(\phi')\},$$

$\ker(p_1) \simeq \ker(\phi') = A'$ so that it is central. It the follows that (E'', p_1) is a central extension of E. By (ii), this central extension is trivial, which means that there is an isomorphism from (E'', p_1) to $(E \times A')$

$$\begin{array}{ccc} E'' & \xrightarrow{p_1} & E \\ {\scriptstyle \delta}\big\downarrow{\scriptstyle \wr} & & \big\| \\ E \times A' & \xrightarrow{\widetilde{p_1}} & E, \end{array} \tag{3.2}$$

where in the previous diagram $\widetilde{p_1}$ is the projection on the first factor and $p_1 = \widetilde{p_1} \circ \delta$ implies $p_1 = p_1 \circ \delta^{-1}$.

Now if $i : E \hookrightarrow E \times A'$ is the homomorphism defined by $i(e) = (e, 1)$ then $\psi \equiv p_2 \circ \delta^{-1} \circ i : E \longrightarrow E'$ is such that

$$\begin{array}{ccc} E & \xrightarrow{\phi} & G \\ {\scriptstyle \psi}\big\downarrow & & \big\| \\ E' & \xrightarrow{\phi'} & G. \end{array}$$

since

$$p_1(\delta^{-1}(e, 1)) \overset{3.2}{=} \widetilde{p_1}(e, 1) \overset{\text{by definition}}{=} e,$$

it follows that

$$\phi(e) = \phi(p_1(\delta^{-1}(e, 1))) \overset{3.1}{=} \phi'(p_2(\delta^{-1}(e, 1))) \overset{\text{definition of } \psi}{=} \phi'(\psi(e)).$$

Since (E', ϕ') was arbitrary and we already showed that morphisms from (E, ϕ) to (E', ϕ') are unique, then (E, ϕ) is a universal central extension of G.

(1) ($\Leftarrow$):

Claim 1. Let $E = [F,F]/[F,R]$, with

$$\phi : [F,F]/[F,R] \longrightarrow [F,F]/R = [F/R, F/R] = [G,G] = G$$

the quotient map. Then (E,ϕ) is a central extension of G.

Justification. Note that R is a normal subgroup of F so that $[F,R]$ is likewise a normal subgroup of F. Let $E_1 = F/[F,R]$. Then, $E \subset E_1$ and E_1 also projects onto G via the quotient map $\phi_1 : F/[F,R] \longrightarrow F/R = G$, and $\phi = \phi_1|E$. Note that $\ker(\phi_1) \subset R/[F,R]$ [1], hence $[E_1, \ker(\phi_1)] \subset [F,R]/[F,R] = 1$[2] $\Longleftrightarrow \ker\phi_1 \subset Z(E_1)$. Thus (E_1,ϕ_1) is a central extension of G, hence (E,ϕ) is also a central extension of G. ($E \subset E_1$ and $\phi_1 = \phi|E$.)

$\square$

Claim 2. (E,ϕ) satisfies (i), i.e., $[E,E] = E$.

Justification. First note that $E = [E_1, E_1]$, thus

$$w \in [E_1, E_1]$$

if and only if

$$w = \prod_{i=1}^{k} [e_i, e_i']^{\delta_i} \text{ for some } e_i, e_i' \in E_1 \text{ and } \delta_i = \pm 1$$

if and only if

$$w = \prod_{i=1}^{k} [f_i[F,R], f_i'[F,R]]^{\delta_i} \text{ for some } f_i, f_i' \in F \text{ and } \delta_i = \pm 1$$

[1] $\ker(\phi_1) = \{x[F,R] \in F/[F,R] \,|\, xR = R\} = \{x[F,R] \in F/[F,R] \,|\, x \in R\} \subset R/[F,R]$.

[2] If $w \in [E_1, \ker\phi_1]$ then $w = \prod_{i=1}^{k} [e_i, k_i]^{\delta_i}$ where $\delta_i = \pm 1$, $e_i \in F/[F,R]$ and $k_i \in R/[F,R]$. Thus $w = \prod_{i=1}^{k} [f_i, r_i]^{\delta_i}[F,R] \in [F,R]/[F,R]$ for some $f_i \in F$ and $r_i \in R$.

if and only if

$$w = \prod_{i=1}^{k} [f_i, f_i']^{\delta_i} [F, R] \text{ for some } f_i, f_i' \in F \text{ and } \delta_i = \pm 1$$

if and only if

$$w \in E.$$

On the other hand, since ϕ is surjective onto G and $\phi_1 = \phi|E$ it follows that every element $e_1 \in E_1$ can be written as $k_1 \cdot e$ for some $k_1 \in \ker(\phi_1), e \in E$. Since $\ker(\phi_1) \subset Z(E_1)$, then $E_1 \subset Z(E_1) \cdot E = E \cdot Z(E_1)$, therefore $E_1 = E \cdot Z(E_1)$. So we obtain

$$E = [E_1, E_1] = [E \cdot Z(E_1), E \cdot Z(E_1)] = [E, E].$$

□

Claim 3. (E, ϕ) satisfies (ii), i.e., all central extensions of E are trivial.

Justification. Let

$$1 \longrightarrow A \longrightarrow E_2 \xrightarrow{\psi} E \longrightarrow 1$$

be any central extension of E. This induces an extension $(E_3, p_1) = (E_1 \times_G E_2, p_1)$ of E_1,

$$\begin{array}{ccccc} E_3 = E_1 \times_G E_2 & & \xrightarrow{\quad p_1 \quad} & & E_1 \\ \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle \phi_1} \\ E_2 & \xrightarrow{\quad \psi \quad} & E & \xrightarrow{\quad \phi \quad} & G, \end{array}$$

where $E_1 \times_G E_2 = \{(x, y) \in E_1 \times E_2 \,|\, \phi_1(x) = (\phi \circ \psi)(y)\}$.
In fact, this is actually a central extension. Indeed, $\ker(p_1) \simeq \ker(\phi \circ \psi)$, since $\ker(p_1) = \{(1, y) \in E_1 \times E_2 \,|\, y \in \ker(\phi \circ \psi)\}$. Now,

$$\psi([E_2, E_2]) = [\psi(E_2), \psi(E_2)] = [E, E] = E,$$

and thus $E_2 = [E_2, E_2] \cdot A$. Also, $\psi([E_2, \ker(\phi \circ \psi)]) \subset [E, \ker(\phi)] = 1$ so that $[E_2, \ker(\phi \circ \psi)] \subset A$. This implies that for $x \in \ker(\phi \circ \psi)$ and $s, t \in$

$E_2, [x,s], [x,t] \in A$ so that $xsx^{-1} = sz_1, xtx^{-1} = tz_2$ for some $z_1, z_2 \in A$. Hence $x[s,t]x^{-1} = [xsx^{-1}, xtx^{-1}] = [sz_1, tz_2] = [s,t]$. Thus x commutes with $[E_2, E_2]$. Since x also commutes with A (A is central), it commutes with all of E_2, and E_3 is a central extension of E_1.

Since F is free we can fill in the above diagram

$$\begin{array}{ccccc} & & & & F \\ & & & \swarrow^{\gamma} & \downarrow{\scriptstyle \alpha} \\ E_3 = E_1 \times_G E_2 & & \xrightarrow{p_1} & & E_1 \\ {\scriptstyle p_2}\downarrow & & & & \downarrow{\scriptstyle \phi_1} \\ E_2 & \xrightarrow{\psi} & E & \xrightarrow{\phi} & G \end{array}$$

where $\gamma : F \longrightarrow E_3$ lifts the quotient map $\alpha : F \longrightarrow E_1$. (Theorem 3.0.7.) This amounts to a homomorphism $\theta : F \longrightarrow E_2$ such that for $x \in F$, $(\phi \circ \psi)(\theta(x))$ coincides with the image of x in $G \simeq F/R$. So $\theta(R) \subset \ker(\phi \circ \psi \subset Z(E_2))$, and

$$\theta([F,R]) \subset [\theta(F), \theta(R)] \subset [E_2, Z(E_2)] = 1.$$

Hence θ descends to $\overline{\theta} : E_1 = F/[F,R] \longrightarrow E_2$ which, together with the identity map on E_1, gives a splitting $(id, \overline{\theta}) : E_1 \longrightarrow E_3 = E_1 \times_G E_2$ of p_1. Restricting to E then gives a trivialization of $\psi : E_2 \longrightarrow E$, verifying (ii).

The final conclusion is that, when G is perfect, the (E, ϕ) from Claim 1 is a universal central extension of G.

(2) ($\Rightarrow$): This follows from uniqueness, since we just constructed a universal central extension satisfying (i) and (ii).

□

■

Proposition 3.0.9 *Let $G = < x_1, \ldots, x_n \,|\, r_1, \ldots, r_m >$ be a finitely presented group, F the free group on $\{x_1, \ldots, x_n\}$ and $R \subset F$ the normal closure of*

$\{r_1, \dots, r_m\}$. *Suppose that G is perfect and choose $c_i \in [F, F]$ such that $x_i c_i \in R$. Then the following is a presentation for the universal central extension of G:*

$$< x_1, \dots, x_n \mid x_i c_i, [x_i, r_j] \, for \, i = 1, \dots, n; j = 1, \dots, m > .$$

Proof. (Bri08, pg.12) We first note that it is possible to choose $c_i \in [F, F]$ such that $x_i c_i \in R$. This follows because $G = F/R$ is perfect and thus $F/R = [G, G] = [F, F]R/R$, which implies $F = [F, F]R$. Thus we conclude that $x_i R = w_i R$ for some $w_i \in [F, F]$ and therefore $x_i w_i^{-1} R = R \iff x_i w_i^{-1} \in R$.

Now, Let $K \subset F$ be the normal closure of the relators in the above presentation. We must prove that F/K is isomorphic to $[F, F]/[F, R]$, the universal central extension of G.

First, we note that $[F, R] \subset K$.

Now, let $\widetilde{X} := \{x_1 c_1, \dots, x_n c_n\}$. Since $x_i c_i \in R$, the image of $\widetilde{X}$, under the canonical projection, in $F/[F, R]$, is central. In particular $K/[F, R]$ is abelian, generated by the image of $\widetilde{X}$.

Since the image of $\widetilde{X}$ generates $F/[F, F]$, the natural map $K/[F, R] \longrightarrow F/[F, F]$ is onto. Moreover, as the image of $\widetilde{X}$ is a basis for $F/[F, F] \simeq \mathbb{Z}^n$, it must also be a basis for $K/[F, R]$. Hence the natural map $K/[F, R] \longrightarrow F/[F, F]$ is an isomorphism. In particular the kernel of this map is trivial, so $K \cap [F, F] \subset [F, R]$. But $[F, R] \subset K$, so $K \cap [F, F] \subset [F, R]$.

Now consider the map $[F, F] \longrightarrow F/K$. As $x_i c_i \in K$ and $c_i \in [F, F]$, the image of this map contains $x_i K$ for $i = 1, \dots, n$. Thus the map is onto. Its kernel is $[F, F] \cap K$, which we just proved is $[F, R]$. Therefore F/K is isomorphic to $[F, F]/[F, R]$.

■

Corollary 3.0.10 *Let G be a perfect group. Then, a central extension (E, ϕ) of G is universal if $H_1(E, \mathbb{Z}) = 0$ and $H_2(E, \mathbb{Z}) = 0$.*

Theorem 3.0.11 *Let G be a perfect group. Then, the kernel of the universal central extension (E, ϕ) of G is naturally isomorphic to $A = H_2(G, \mathbb{Z})$.*

4

Computability

In this chapter we want to fix our notion and terminology regarding the theory of computability and related notions. In this vain, our informal notion of a procedure can be identified with the formal notion of a Turing Machine. A fundamental concept in the theory of computability is that of recursively enumerable set, which we try to capture in the following definition.

Definition. (i) A subset $A \subset \mathbb{N}$ **is Recursively Enumerable** if there is a Turing Machine that stops only when run on a tape containing the representation of a member of A. Informally, this means that there is a procedure that can *potentially enumerate* all the elements of A and only the elements of A.

(ii) A subset $A \subset \mathbb{N}$ **is Recursive** if both A and $\mathbb{N} \setminus A$ are recursively enumerable. Informally, this means that there is a procedure that can decide, for any $n \in \mathbb{N}$, in finitely many steps, whether or not n is an element of A.

Remark. Even though the previous definitions apply to subsets $A \subset \mathbb{N}$ it can be naturally extended to any class of objects that can be codified by natural numbers.

Using the previous terminology we can describe informally what it means to solve a problem $\mathcal{P}$ algorithmically. Initially, the problem $\mathcal{P}$ consists of a class of "positive instances". These instances can be coded by natural numbers and the corresponding class of codes forms a set $A \subset \mathbb{N}$. To solve the problem $\mathcal{P}$ is to find an algorithm recognizing the set A. In other words, the problem $\mathcal{P}$ is solvable if and only if the set A is recursive.

The first problem known to be effectively unsolvable was intimately connected with the existence of a set $A \subset \mathbb{N}$ which is recursively enumerable but not recursive. Such a problem came to be known as the **Halting problem** and its unsolvability was established by Alan Turing in 1937. Formally, the Halting problem asks whether there is a Turing machine that can decide, for any other Turing machine M, whether or not M stops when started on an empty tape.

Theorem 4.0.12 (A. Turing, 1937.) *The Halting problem is algorithmically unsolvable.*

4.1 Undecidable Problems in Group Theory

Motivated by topology, Max Dehn, in his attempts to understand low dimensional manifolds, posed in 1910 three fundamental questions of combinatorial group theory. Assuming that the groups involved are finitely presented, these questions can be stated as follows:

1. **The word problem.** Is there an algorithm to recognize the identity of a group? More precisely:

   Given a word w in the generators of G, does w represent 1 in G?

2. **The conjugacy problem.** Is there an algorithm to decide whether two given elements of a group are conjugate? In more detail:

   Given words w_1 and w_2 in the generators of G, do w_1 and w_2 represent conjugate elements of G?

3. **The Isomorphism Problem** Is there an algorithm to decide whether two given groups are isomorphic? Being precise:

   Can we find a procedure that, given any two finitely presented groups G and H, can decide whether or not they are isomorphic?

There are many classes of groups for which the word problem is decidable. For instance, finite groups, finitely presented abelian groups, and free groups on finitely many generators. Nevertheless, P.S. Novikov, (Nov58), and W. Boone (Boo59) independently proved that there is a finitely presented group for which the word problem is unsolvable.

Theorem 4.1.1 (Novikov-Boone-Britton, 1954-1958.) *There is a finitely presented group G that has an algorithmically unsolvable word problem.*

The analogue for finitely presented semigroups had been proved earlier, by E. Post and A. Markov (Rot94, p. 428).

Theorem 4.1.2 (Markov-Post, 1947.) *There is a semi-group G that G has algorithmically unsolvable word problem.*

The proofs of these results are based on a natural reduction to the Halting problem. For further details on the unsolvability of these problems, and also for a self contained treatment on computability, we recommend (Rot94) Chapter 12. Finally we mention that the unsolvability of the word problem implies directly the corresponding unsolvability of the conjugacy problem.

4.1.1 The Triviality Problem

The remaining decision problem of combinatorial group theory is the fundamental **Triviality Problem**, which asks whether there is a procedure to decide the triviality of any finitely presented group G. As in the previous cases, this problem can be reduced to the solution of the word problem. But in fact, something even more general is true, as we describe next.

Definition. An algebraic property of finitely presented groups, i.e. a property that is preserved under isomorphism, is called a **Markov property** if it satisfies the following conditions:

(i) there is a finitely presented group G_+ with the property,

(ii) there is a finitely presented group G_- which cannot be embedded in a group with the given property.

There are many examples of Markov properties. Some examples are triviality, finiteness, abelianess, simplicity, freeness, and having solvable word problem. Using the undecidability of the word problem, Adian (Ady57) and Rabin (Rab58) proved, independently, the following fundamental result. (We follow the proof sketch in the exposition paper (And05).)

Theorem 4.1.3 (Adian-Rabin, 1957-1958.) *Let $\mathcal{P}$ be a Markov property. Then there is no algorithm which decides whether or not any finitely presented group has the property $\mathcal{P}$.*

Proof. Fix a Markov property $\mathcal{P}$, as well as groups G_+ and G_- showing $\mathcal{P}$ to be a Markov property, and G, a group with unsolvable word problem. We will construct a recursive sequence of groups H_w for each word $w \in G$. This will be done in such a way that H_w **has the property $\mathcal{P}$ if and only if w is trivial in** G. In order to do this, we will use the following technical algebraic lemma whose proof can be found in (Mil90).

Lemma 4.1.4 *Let K be a group with a specified finite presentation*

$$K = \langle X \mid R \rangle,$$

$X = \{x_1, \ldots x_n\}$. Fix a word w in the generators of K. Let S be the following set of relations

$$a^{-1}ba = c^{-1}b^{-1}cbc$$

$$a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2$$

$$a^{-3}\,[w, b]\,a^3 = c^{-3}bc^3$$

$$a^{-(3+i)}x_iba^{(3+i)} = c^{-(3+i)}bc^{(3+i)},\ i = 1, \ldots, n,$$

where $[w, b]$ is the commutator. Next, define

$$L_w = \langle \{a, b, c\} \cup X \mid S \cup R \rangle.$$

Then we have the following:

(i) L_w is generated by two elements: b and ca^{-1},

(ii) If $w = 1$ in K, then L_w is trivial,

(iii) If $w \neq 1$ in K, then K is embedded in L_w via the inclusion of generators.

Assuming the lemma, we will generate the sequence of groups H_w as follows: Let

$$K = G * G_-$$

be the co-product of G with G_-. The group K it is generated by the disjoint union of the relation-sets for G and G_-. Then its presentation can be written easily. (The fact that the co-product remains in the range of finitely presented groups is very important.) Now, for any word $w \in G$, consider w as a word in K and construct L_w as in the lemma. Then, let

$$H_w = L_w * G_+.$$

This is again a co-product construction. Thus, all along, the construction could easily have been done by a machine with input w and with access to the finite presentation of G, G_- and G_+. We see that if $w = 1 \in G$ then $w = 1 \in K$ and L_w is trivial. If $w \neq 1$ then $w \neq 1 \in K$, and K embeds in L_w. Therefore, K embeds in H_w. But G_- embeds in K. This yields G_- in H_w, which implies H_w does not have the property $\mathcal{P}$. Thus, a recursive way to decide whether H_w has the property $\mathcal{P}$ would solve the word problem for G, which is impossible.

■

As a consequence of the previous theorem it follows that the triviality problem is unsolvable. In then follows immediately the unsolvability of the isomorphism problem.

Theorem 4.1.5 *The isomorphism problem for finitely presented groups is unsolvable.*

Proof. Since being trivial is a Markov property, it is undecidable whether or not a group is isomorphic to the trivial group. But the triviality problem is a subproblem of the isomorphism problem, so the result follows.

■

We next introduce a very important related concept which plays a decisive role in the proof of the main unsolvability result of this monograph.

Definition. A recursive family of finitely presented groups is an **Adian-Rabin sequence** if there is no algorithm to check whether an arbitrary element of the sequence represents the trivial group.

Definition. An Adian-Rabin sequence whose elements have trivial first and second homology groups, will be called a **Novikov sequence**.

The final result of this section has to do with the technical fact that given a finitely presented perfect group, there is a procedure to effectively construct a finite presentation of its universal central extension.

Proposition 4.1.6 *There exists an algorithm that, given a finite presentation* $< X \mid \Sigma >$ *of a perfect group* G*, will output a finite presentation* $< X \mid \widetilde{\Sigma} >$ *for the universal central extension of* G*. (See Proposition 3.0.9.)*

Proof. (Bri08, pg.13) Let F be the free group on $X = \{x_1, \dots x_n\}$. We start by pointing out the following facts:

(i) It is possible to enumerate the elements of F effectively. The enumeration proceeds by stages: at stage k, all the words of length k are generated systematically, for example according to the lexicographic order.

(ii) It is possible to enumerate the elements $d_0, d_1, \dots$ of $[F, F]$ effectively. At stage k, we generate the words of length $2k$ containing the words generated at stage k in (i). (It is known that, for any group G, its commutator subgroup is the set $[G, G] = \{a_1 a_2 \cdots a_n \cdot a_1^{-1} a_2^{-1} \cdots a_n^{-1} \mid a_i \in G \text{ and } n \geq 2\}$.)

(iii) It is possible to enumerate the elements $\rho_0, \rho_1, \dots$ of the normal closure of Σ. At stage k the procedure generates all the words of length k of the form $\prod_{i=1}^{k} f_i r_i f_i^{-1}$, where f_i is a word generated at stage k of (i) and r_i is a relator.

(iv) It is possible to enumerate effectively the elements of the form (d_i, ρ_j), where $d_i \in [F, F]$ and ρ_j belongs to the normal closure of Σ. At stage k the procedure generates all pairs involving d_0 to d_n and ρ_0 to ρ_n.

Now, since G is perfect, it follows from the facts mentioned above that there exists an effective procedure to enumerate the elements of $\widetilde{\Sigma}$:
The procedure generates systematically the list of all pairs (d_i, ρ_j) and for each $x \in X$, in turn, it runs through the products $xd_i\rho_j$ to check whether any of them is equal to the identity in F (that is, freely equal to the empty word). Since the given group G is perfect, the procedure eventually finds indices $i(x)$ and $j(x)$ such that $xd_{i(x)}\rho_{j(x)}$ is equal to the identity in F. Finally, the main algorithm that we seek outputs

$$\widetilde{\Sigma} = \{xd_{i(x)} \,|\, x \in X\} \cup \{[\sigma, x_k] \,|\, \sigma \in \Sigma, x \in X\}.$$

■

4.2 Semi-algebraic Sets

Since our purpose is to prove a smooth version of Markov's Theorem we will introduce in this section some basic concepts from Real Algebraic Geometry. Our fundamental tool will be the famous Tarski-Seidenberg Thorem. The main references here are (Mar08), (Cos00), and (BPCR06).

Definition. Let R be a real closed field. If $\mathcal{P}$ is a finite subset of $R[x_1, \ldots, x_k]$, we denote **the set of zeros of** $\mathcal{P}$ in R^k as

$$Zer(\mathcal{P}, R^k) = \left\{ x \in R^k \,\middle|\, \bigwedge_{P \in \mathcal{P}} P(x) = 0 \right\}.$$

These are the **algebraic sets** of $R^k = Zer(\{0\}, R^k)$.

Definition. The family of **semi-algebraic sets** of R^k is the smallest class containing the algebraic sets, as well as the sets of the form $\left\{x \in R^k \,|\, P(x) > 0\right\}$ with $P \in R[X_1, \ldots, X_k]$, and which is closed under boolean operations (complementation, finite unions, and finite intersections).

Remark. Any semi-algebraic set in R^k is the finite union of sets of the form

$$\left\{ x \in R^k \,|\, P(x) = 0 \wedge \bigwedge_{Q \in Q} Q(x) > 0 \right\}.$$

These are the **basic semi-algebraic sets**.

We give now some examples of semi-algebraic sets.

Example. (i) The semi-algebraic sets of $\mathbb{R}$ are the union of finitely many points and open intervals.

(ii) An algebraic subset of $\mathbb{R}^n$, defined by polynomial equations, is semi-algeraic.

(iii) If $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are semi-algebraic, then $A \times B$ is a semi-algebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$.

Definition. Let $S \subset R^k$ and $T \subset R^l$ be semi-algebraic sets. A function $f : S \longrightarrow T$ is **semi-algebraic** if its graph Graph(f) is a semi-algebraic subset of R^{k+l}.

We give now some examples of semi-algebraic functions.

Example. (i) If $f : A \longrightarrow B$ is a polynomial mapping, i.e., all its coordinates are polynomial, it is semi-algebraic.

(ii) If $f : A \longrightarrow \mathbb{R}$ is a semi-algebraic function, then $|f|$ is semi algebraic. Furthermore, if $f \geq 0$ on A, then $\sqrt{f}$ is semi-algebraic

Proposition 4.2.1 (i) *The direct image and the inverse image of a semi-algebraic set by a semi-algebraic mapping are semi-algebraic.*

(ii) *The composition of two semi-algebraic mappings is semi-algebraic.*

Definition. A **non-singular algebraic hypersurface** is the zero set $Zer(Q, R^k)$ of a polynomial $Q \in R[x_1, \ldots, x_k]$ such that the **gradient** of Q, i.e. the vector

$$Grad(Q)(p) = \left(\frac{\partial Q}{\partial x_1}(p), \ldots, \frac{\partial Q}{\partial x_k}(p)\right)$$

is never 0 for $p \in Zer(Q, R^k)$.

We will now introduce our main tool in this section.

We consider systems of polynomial equations and inequalities of the form

$$S(X) : \begin{cases} f_1(X) \rhd_1 0 \\ \vdots \\ f_k(X) \rhd_k 0 \end{cases}$$

where $\rhd_i \in \{\geq, >, =, \neq\}$ and each $f_i(X)$ is a polynomial in n variables $X_1, \ldots, X_n$ with coefficients in $\mathbb{Q}$.

Theorem 4.2.2 (Tarski-Seidenberg) *Given a system of polynomial equations and inequalities $S(T, X)$ in $m + n$ variables $T_1, \ldots, T_m, X_1, \ldots, X_n$ with coefficients in $\mathbb{Q}$, there exist finitely many systems of polynomial equations and inequalities $S_1(T), \ldots, S_l(T)$, with coefficients in $\mathbb{Q}$, such that, for each real closed field R and each $t = (t_1, \ldots, t_m) \in R^m$, the system $S(t, X)$ has a solution $x = (x_1, \ldots, x_n) \in R^n$ if and only if t is a solution of one of the systems $S_1(T), \ldots, S_l(T)$.*

Remark. There is a general procedure which computes the systems $S_1(T), \ldots, S_l(T)$, in terms of the system $S(T, X)$.

Now, we specify what it is meant by a first-order formula in the language of real closed fields. A **first-order formula** is a formula obtained by the following constructions:

1. If $f \in Q[X_1, \ldots, X_n], n \geq 1$, then $f \geq 0, f > 0, f = 0$, and $f \neq 0$ are first-order formulas.

2. If Φ and Ψ are first-order formulas, then the following are first-order formulas as well, Φ **and** Ψ**,** Φ **or** Ψ**,** and **not** Φ. These are often denoted by $\Phi \vee \Psi, \Phi \wedge \Psi$ and $\neg\Phi$ respectively.

3. If Φ is a first-order formula then $\exists X \Phi$ and $\forall X \Phi$ are first-order formulas.

Those formulas obtained using only constructions 1 and 2 are called **quantifier free formulas**.

Definition. We say that two first-order formulas $\Phi(X_1, \ldots, X_n)$ and $\Psi(X_1, \ldots, X_n)$ are equivalent if for every real closed field R and every $x \in R^n$, $\Phi(x)$ holds in R if and only if $\Psi(x)$ holds in R.

Theorem 4.2.3 (Tarski-Seidenberg, General Form.) *Every first-order formula in the language of real closed fields is equivalent to a quantifier-free formula,i.e., the language of real closed fields admits elimination of quantifiers.*

Proof. See Appendix I of (Mar08).

■

4.3 Finite presentation of differentiable and combinatorial manifolds

At this point, it becomes necessary to discuss the technical detail of how to represent, in a finite notation, a given combinatorial or differentiable manifold. An extended discussion of this topic, along with a very plausible solution, appears in (BHP68), Section 3.2, whose treatment we adopt without reservations.

The contention is that a finite presentation $\mathcal{M}$ of an n-manifold M, should satisfy the following conditions:

(i) $\mathcal{M}$ is a finite sequence of symbols in some language,

(ii) there is an algorithm to determine whether any given finite notation in this language represents a manifold,

(iii) associated with each presentation $\mathcal{M}$, there is precisely one n-manifold $M(\mathcal{M})$, represented by $\mathcal{M}$.

(iv) the notation $\mathcal{M}$ represents $M(\mathcal{M})$ in a "natural" way.

As to the interpretation of (iv), the point of view adopted in (BHP68), is that a finite presentation $\mathcal{M}$ of a differentiable and combinatorial manifold should have the property that a C^∞ atlas $\mathcal{U}$ of $M(\mathcal{M})$, and a corresponding compatible triangulation Δ, should be described by $\mathcal{M}$.

One way to fulfill the previous requirements is the adoption of the notion of an **Algebraic Atlas Presentation** $\mathcal{M}$, associated with a given manifold $\hat{M}$. This notation allows the effective recovery of either the combinatorial or differential structure of the given manifold $\hat{M}$. The corresponding formal statement is as follows:

Theorem 4.3.1 ((BHP68), Section 3.2, Theorem 4.) *For every closed differentiable n-manifol $\hat{M}$ there exists (a finite) algebraic atlas presentation $\mathcal{M}$ such that the manifold $M(\mathcal{M})$ presented by $\mathcal{M}$ is diffeomorphic to $\hat{M}$. Moreover, the concept of algebraic presentation fulfills the requirements (i), (ii),(iii) stated at the beginning of this section, and corresponding to (iv) the following:*

(iv′) *if an algebraic atlas presentation $\mathcal{M}$ is given, then the corresponding C^∞-atlas $\mathcal{U}(\mathcal{M})$ presented by $\mathcal{M}$ can be recursively computed in a natural way.*

4.4 From Adian-Rabin to Novikov sequences

Suppose that $\prod = \{\pi_i\}_{i\in\mathbb{N}}$ is an Adian-Rabin sequence. Since there is an algorithm that checks whether $H_1(\pi_i, \mathbb{Z})$ is 0, (see, for example, (Mil90)) we can construct a new recursive sequence

$$\prod{}' = \{\pi \in \prod \mid H_1(\pi, \mathbb{Z}) = 0\}\,..$$

Since the trivial group π_0 has $H_1(\pi_0, \mathbb{Z}) = 0$, it follows that the property of being a trivial element in $\prod'$ is not algorithmically recognizable. We then conclude that $\prod'$ is indeed an Adian-Rabin sequence.
Now from $\prod'$ we will construct a Novikov sequence, using the following result:

Theorem 4.4.1 *Given a finite presentation of a group π, with $H_1(\pi, \mathbb{Z}) = 0$, one can effectively construct a presentation of a new group $\widetilde{\pi}$, with a central extension*

$$1 \longrightarrow H_2(\pi, \mathbb{Z}) \longrightarrow \widetilde{\pi} \longrightarrow \pi \longrightarrow 1, \tag{4.1}$$

such that $H_1(\widetilde{\pi}, \mathbb{Z}) = H_2(\widetilde{\pi}, \mathbb{Z}) = 0$.

Proof. Let $\langle h_1, \ldots, h_k \mid q_1, \ldots, q_m\rangle$ be a presentation of the group π and consider the exact sequence

$$1 \longrightarrow R \longrightarrow F \longrightarrow \pi \longrightarrow 1, \tag{4.2}$$

where F is the free group on the generators $h_1, \ldots, h_k$ of π and R is the normal closure of the relators $q_1, \ldots, q_m$. Since $H_1(\pi, \mathbb{Z}) = 0 \iff \pi/[\pi, \pi] = 0$ it follows that π is perfect. Then by Theorem 3.0.8 it has a universal central extension given by $\widetilde{\pi} : [F,F]/[F,R] \longrightarrow \pi$. Furthermore, Theorem 3.0.11 implies that the kernel of $\widetilde{\pi} : [F,F]/[F,R] \longrightarrow \pi$ is $[F,F] \cap R/[R,F] = H_2(\pi,\mathbb{Z})$ (by Hopf's formula, (Hop42)) and we thus obtain the exact sequence

$$1 \longrightarrow H_2(\pi,\mathbb{Z}) \longrightarrow \widetilde{\pi} \longrightarrow \pi \longrightarrow 1.$$

On the other hand, Corollary 3.0.10 asserts that $H_1(\widetilde{\pi},\mathbb{Z}) = H_2(\widetilde{\pi},\mathbb{Z}) = 0$.

The effectiveness of the construction of $\widetilde{\pi}$ follows from Corollary 4.1.6.

■

Finally we note that by applying the previous theorem to the modified sequence $\prod'$ we obtain a new Adian-Rabin sequence:

$$\widetilde{\prod} = \left\{ \widetilde{\pi} \,|\, \pi \in \prod' \right\}.$$

Theorem 4.4.2 *$\widetilde{\prod}$ is, in fact, a Novikov sequence.*

Proof. If $\pi = 0$, $H_2(\pi,\mathbb{Z}) = 0$. Therefore, in the central extension 4.1, we have

$$1 \longrightarrow 0 \longrightarrow \widetilde{\pi} \longrightarrow 0 \longrightarrow 1$$

and thus $\widetilde{\pi} = 0$. Now, if $\widetilde{\pi} = 0$ in the central extension 4.1, we have

$$1 \longrightarrow H_2(\pi,\mathbb{Z}) \longrightarrow 0 \longrightarrow \pi \longrightarrow 1,$$

but the exactness of this sequence implies that $\pi = 0$. Therefore $\pi = 0$ if and only if $\widetilde{\pi} = 0$. Since there is no algorithm that can decide whether $\pi \in \prod'$ is the trivial group, there is similarly no algorithm for $\widetilde{\pi} \in \widetilde{\prod}$.

■

5

Superperfect Groups and Homology Spheres

In this chapter we finally present the main result of this monograph, whose proof hinges on the intimate connection, discovered by M.A. Kervaire, between some very special groups (the superperfect groups) and the so-called homology spheres. These groups are special because their first and second homology groups are trivial. We will start with a such a group π and show how to construct effectively a compact non-singular algebraic hypersurface $S \subset \mathbb{R}^{n+1}$ so that S is a homology sphere and $\pi_1(S) = \pi$. Moreover we will do this in such a way that S is diffeomorphic to $\mathbb{S}^n$ if and only if π is trivial. (Corollary 2.2.16.)

The idea is to use the so-called Dehn construction [1] to generate a 2-dimensional simplicial complex $\mathcal{K}$ such that $\pi_1(\mathcal{K}) = \pi$. In the next step one "extends" $\mathcal{K}$ to a regular neighbourhood N in $\mathbb{R}^{n+1}$, and then smooths out the corners of this neighbourhood. The result is a hypersurface $Q \subset \mathbb{R}^{n+1}$ obtained by taking the boundary of the smoothed out neighbourhood N mentioned above. Such a hypersurface is a compact n-dimensional manifold Q with fundamental group π. The explicit details of this construction can be found in (BHP68), Lemma 6, Section 3.3. Even more, it can be shown using the Poincaré Duality Theorem 2.2.4, that the homology groups of Q are trivial, except for the second and the $(n-2)$th, which are the direct sum of several copies of $\mathbb{Z}$.

[1] For details see (Mau70), Theorem 3.3.20

In fact, it can be proved that $H_{n-2}(Q;\mathbb{Z}) = H_2(Q;\mathbb{Z})$. We start by noting that, by Theorem 2.2.5, $H_n(Q) = \mathbb{Z}$ because $Q \subset \mathbb{R}^{n+1}$ is a hypersurface and therefore orientable. Furthermore, using Proposition 2.2.2, we obtain

$$H^2(Q) \approx FH_2(Q;\mathbb{Z}) \oplus TH_1(Q;\mathbb{Z}).$$

But as we mentioned above $H_2(Q;\mathbb{Z})$ is free abelian and therefore

$$FH_2(Q;\mathbb{Z}) = H_2(Q;\mathbb{Z}),$$

consequently, applying Poincaré Duality Theorem we finally obtain

$$H_{n-2}(Q;\mathbb{Z}) = H^2(Q) = FH_2(Q;\mathbb{Z}) \oplus \cancelto{0}{TH_1(Q;\mathbb{Z})} = H_2(Q;\mathbb{Z}).$$

In the next stage of the construction one realizes all generators of the second homology group of Q by imbedded 2-spheres and kills them by surgeries. Once all generators of the second homology group of Q will be killed, we must smoothed out the corners. The result will be a compact hypersurface S which is the smoothed out boundary of a tubular neighbourhood of a finite 3-dimensional acyclic complex $\overline{\mathcal{K}}$ imbedded in $\mathbb{R}^{n+1}$ and such that $\pi_1(\overline{\mathcal{K}}) = \pi$.

5.1 Computing Normal Bundles

Let's assume then that we already have the hypersurface Q with the stated properties mentioned above, then by virtue of H. Hopf's Theorem 2.2.8, and the fact that $H_2(\pi;\mathbb{Z})$ is trivial, we obtain the exact sequence

$$\pi_2(Q) \xrightarrow{\rho} H_2(Q;\mathbb{Z}) \longrightarrow H_2(\pi;\mathbb{Z}) \longrightarrow 0.$$

Now, the First Isomorphism Theorem for groups implies that

$$H_2(\pi;\mathbb{Z}) \simeq H_2(Q;\mathbb{Z})/\rho\pi_2(Q).$$

But $H_2(\pi;\mathbb{Z}) = 0$ and therefore the Hurewicz homomorphism ρ is surjective. Thus, all generators of $H_2(Q;\mathbb{Z})$ can be represented by continuous functions from $\mathbb{S}^2$ into Q. Furthermore, by Whitney's imbedding Theorems 2.2.17 we can find an imbedding $\overline{f} : \mathbb{S}^2 \longrightarrow Q$ such that $\overline{f} \in \pi_2(Q)$ and $\overline{f}$ is homotopic to f. (Theorem 2.2.17.) Hence all generators of $H_2(Q;\mathbb{Z})$ can be realized by imbedded spheres. Now Theorem 2.2.10 guarantees that these generators can be represented by non-intersecting imbedded spheres. Next we will prove that these spheres satisfy a very special condition.

Claim 1. The imbedded spheres realizing the generators of $H_2(Q;\mathbb{Z})$ have trivial normal bundles.

Justification. Let's start by pointing out that any homotopy class of functions from $\mathbb{S}^2$ to Q contains an imbedding

$$f : \mathbb{S}^2 \longrightarrow Q.$$

(See (Whi36), Theorem 2.) In the sequel, we use the following notation:
τ_Q^n will denote the tangent bundle of Q, $f^*\tau_Q^n$ the restriction of this bundle to $\mathbb{S}^2$, similarly $\tau_{\mathbb{S}^2}$ will denote the tangent bundle of $\mathbb{S}^2$, and finally ϵ^k is the trivial k-dimensional vector space bundle over $\mathbb{S}^2$. We know that the induced bundle $f^*\tau_Q^n$ over $\mathbb{S}^2$ splits as the Whitney sum of a sub-bundle isomorphic to $\tau_{\mathbb{S}^2}$ and a complementary sub-bundle ν_f, that is,

$$f^*\tau_Q^n \simeq \tau_{\mathbb{S}^2} \oplus \nu_f. \tag{5.1}$$

Now the fact that $\epsilon^1 \oplus \tau_{\mathbb{S}^2}$ is trivial, follows from the next two facts:

1. The normal bundle of $\mathbb{S}^2 \in \mathbb{R}^3$ is isomorphic to the product bundle $\epsilon^1 = \mathbb{S}^2 \times \mathbb{R}$ by the map $(x, tx) \mapsto (x, t)$.

2. We can identify ϵ^1 with the normal bundle of $\mathbb{S}^2$ in $\mathbb{R}^3$. This way, we can think of $\tau_{\mathbb{S}^2} \oplus \epsilon^1$ as $\tau_{\mathbb{S}^2} \oplus \nu^2$, where ν^2 denotes the normal bundle of $\mathbb{S}^2$ in $\mathbb{R}^3$. On the other hand, we can prove that $\tau_{\mathbb{S}^2} \oplus \nu^2$ is the trivial product bundle $\epsilon^3 = \mathbb{S}^2 \times \mathbb{R}^3$ as follows: first, the elements of the Whitney sum are quintuples $(x, x, v, x, tx) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}^3$ with $x \perp v$, and second, the map $(x, x, v, x, tx) \mapsto (x, v + tx)$ gives an isomorphism of $\tau_{\mathbb{S}^2} \oplus \nu^2$ with $\mathbb{S}^2 \times \mathbb{R}^3$.

Next, to prove that ν_f is trivial, we proceed as follows:

1. $f^*\tau_Q^n$ is trivial (since Q is s-parallelizable, see Proposition 2.3.6) which simply means that $f^*\tau_Q^n \simeq \mathbb{S}^2 \times \mathbb{R}^n = \epsilon^n$ wich by the decompositin in 5.1 implies $\tau_{\mathbb{S}^2} \oplus \nu_f \simeq \mathbb{S}^2 \times \mathbb{R}^n = \epsilon^n$.

2. $\epsilon^1 \oplus \tau_{\mathbb{S}^2} \simeq \epsilon^3$.

Finally, the proof proceeds according to the following statements:

$$\tau_{\mathbb{S}^2} \oplus \nu_f \simeq \epsilon^n$$

if and only if

$$\epsilon^1 \oplus (\tau_{\mathbb{S}^2} \oplus \nu_f) \simeq \epsilon^1 \oplus \epsilon^n$$

if and only if

$$(\epsilon^1 \oplus \tau_{\mathbb{S}^2}) \oplus \nu_f \simeq \epsilon^{n+1}$$

if and only if

$$\epsilon^{2+1} \oplus \nu_f \simeq \epsilon^{n+1}.$$

And thus from Lemma 2.3.3 we conclude the triviality of ν_f.

□

5.2 Constructing the generators of $H_2(Q;\mathbb{Z})$

Now that we have established the existence of these special imbedded spheres, our task is to prove that they can be effectively found by a trial an error algorithm, as we sketch next:

Claim 2. There is a procedure that finds imbedded spheres (in Q) realizing the generators of $H_2(Q;\mathbb{Z})$.

Justification. First we must emphasize that the existence of these disjoint spheres is guaranteed by the previous discussion. The issue at hand is establishing their existence algorithmically.

From the description of Q we have at our disposal a corresponding triangulation and thus, the finitely many generators of $H_2(Q;\mathbb{Z})$ can be represented by associated simplicial chains. In order to realize these generators we look for a collection of disjoint polynomial imbeddings

$$f : \mathbb{S}^2 \longrightarrow N'(Q),$$

where $N'(Q)$ is a sufficiently small open neighbourhood of Q, for example determined by half the injectivity radius of the normal exponential map for Q. For any such imbedding, we check whether its orthogonal projection to Q is an imbedded sphere. We notice that the determination of the injectivity radius and the normal vectors to Q, are semi-algebraic operations, see for example (Nab96, p. 15), and since Q along with f are semi-algebraic, this requirement, along with the disjointness of all the imbedded spheres, constitute a semi-algebraic condition and thus, by Tarski-Seidenberg, it can be checked algorithmically, for every fixed list of rational numbers representing the components of the polynomial mapping f. It then follows that we can generate these polynomial imbeddings systematically, verifying each time whether it represents an imbedded sphere realizing one of the generators of $H_2(Q;\mathbb{Z})$.

□

5.3 Killing $H_2(Q;\mathbb{Z})$ by surgery

Now that we have proved that these spheres have trivial normal bundles and that have been effectively found, we are going to kill one by one these generators by surgeries, and we would like to perform these surgeries inside $\mathbb{R}^{n+1}$.

At the beginning, the closed unbounded component $\overline{U}$ of the complement of Q is a deformation retract of $\mathbb{R}^{n+1} \setminus \mathcal{K}$, since by the Collar Theorem, the space $N \setminus \mathcal{K}$ is homeomorphic smoothly to $\partial N \times [0\,,1)$. But using general position, as in the proof of Lemma 3.2 in (PV12), it can be proved that the pointed space $(\mathbb{R}^{n+1} \setminus \mathcal{K}, *)$ is 2-connected, that is, $\mathbb{R}^{n+1} \setminus \mathcal{K}$ is 2-connected. Thus any imbedded sphere $\sigma : \mathbb{S}^2 \longrightarrow Q \subset \overline{U}$, realizing a generator of $H_2(Q;\mathbb{Z})$, will be null homotopic in $\overline{U}$. If $n+1 \geq 7$ then, by Corollary 15.7 (Bre93), we can realize this homotopy by a 3-disc imbedded in $\overline{U}$, meeting Q transversally along σ. On the other hand, if $n+1 = 6$, then one must also apply the Whitney trick to get the corresponding

imbedded 3-disc. This guarantees that we can perform the first surgery inside $\mathbb{R}^{n+1}$, generating a corresponding space Q_1.
Once we iterate this process and perform j surgeries, the corresponding effect of these surgeries is a new space Q_j. To show that the next surgery can be done inside $\mathbb{R}^{n+1}$ we need to prove that the first and second homotopy groups of the outer connected component $\overline{U}_j$ of the complement of Q_j are trivial. But Q_j is the boundary of a tubular neighbourhood of a 3-dimensional complex $\mathcal{K}_j$ imbedded in $\mathbb{R}^{n+1}$. Thus, $\overline{U}_j$ is homotopy equivalent to the complement of $\mathcal{K}_j$. If $n+1 \geq 7$, this implies that $\overline{U}_j$ is 2-connected. If $n+1=6$, this immediately implies that $\overline{U}_j$ is simply connected. In order to show that $\pi_2(\overline{U}_j)$ is trivial, we first note the following: since $\overline{U}_j$ is simply connected and therefore 1-connected, by the Hurewicz Theorem we obtain the equality $\pi_2(\overline{U}_j) = H_2(\overline{U}_j;\mathbb{Z})$. Next we observe that $H_2(\mathcal{K};\mathbb{Z})$ is free abelian [1] and $\mathcal{K}_j$ is obtained from $\mathcal{K}$ by adding 3-cells killing several linearly independent generators of $H_2(\mathcal{K};\mathbb{Z})$, and thus we obtain that $H^3(\mathcal{K}_j)$ is trivial. Now it olny remains to apply the Alexander Duality Theorem.

Once all the required surgeries have been performed and all generators of the second homology group of Q will be killed, we obtain a new hypersurface Q' whose corners must be smoothed out. (See for example (Cai61)). This smoothing process generates a new hypersurface S, which is the boundary of a small neighbourhood of a finite 3-dimensional acyclic complex $\overline{\mathcal{K}}$, whith $\pi_1(\overline{\mathcal{K}}) = \pi$. It follows from the Collar Theorem and Lemma 2.2.18 that the fundamental group of the constructed hypersurface is isomorphic to $\pi_1(\overline{\mathcal{K}})$, and thus to π.

The final conclusion is that the homology of Q' (and similarly that of S) coincides with that of an n-sphere, consequently the non-singular algebraic hypersurface S is a homology n-sphere.

5.4 Smoothing out effectively

Finally, all that remains is to verify that the previous construction is effective, which is the substance of the following statement.

[1] If X is an n-dimensional CW complex, then $H_n(X)$ is free.

Claim 3 The smoothing out of Q', which generates the hypersurface S, can in fact be performed effectively.

Justification. We describe how to perform the smoothing of the corners on the last stage above. What we want is to find a polynomial

$$p(x_1, \ldots, x_{n+1}) = \sum_{i=(i_1,\ldots,i_{n+1})} T_i x_1^{i_1} \cdots x_{n+1}^{i_{n+1}} \in \mathbb{Q}[x_1, \ldots, x_{n+1}]$$

whose gradient does not vanish at any point of its zero set $Z(p)$ and so that p generates a non-singular hypersurface S approximating the piecewise smooth hypersurface Q'.

Specifically, let $r(Z(p))$ denote the injectivity radius of the normal exponential map for $Z(p)$. We require that on the normal to every point $x \in Z(p)$ there exists a single point $y(x) \in Q'$ such that $\|x - y(x)\| < \frac{r(Z(p))}{2}$ and the map

$$\begin{array}{rccc} h: & Z(p) & \longrightarrow & Q' \\ & x & \mapsto & y(x) \end{array}$$

be a homeomorphism.

Next we show that this condition can be written as a first-order formula of the theory of real closed fields. For convenience, we use the notation

$$\phi(z, u) \equiv (z - u) \parallel \nu(u)$$

where $\nu(u)$ denotes the unit normal to $Z(p)$ at u. Now, we formalize the following conditions:

1. The correspondence h is a function from $Z(p)$ into Q':

$$\forall x \in Z(p) \exists! y(x) \in Q' \, \phi(y(x), x).$$

2. h is injective:

$$\forall u, v \in Z(p) \forall w \in Q' \left[\phi(w, u) \wedge \phi(w, v) \Longrightarrow u = v\right].$$

3. h is a homeomorphism (from 1 and 2 it suffices to represent the continuity of $y(x)$, since $Z(p)$ is compact):

$$[\forall x \in Z(p) \, \forall \epsilon > 0 \, \exists \delta > 0 \, \forall x' \in Z(p)]$$

$$[\|x - x'\| < \delta \Longrightarrow \forall u, v \in Q' \; (\phi(u, x) \wedge \phi(v, x') \Longrightarrow \|u - v\| < \epsilon)] .$$

But the outer unit normal ν is a semi-algebraic function of $Z(p)$ and the injectivity radius $r(Z(p))$ of the normal exponential map for $Z(p)$ is a semi-algebraic function of the coefficients of p. It then follows, from the properties in section 4.2, that the formalization of our requirement generates a semi-algebraic condition. By virtue of the Tarski-Seidenberg Theorem this semi-algebraic condition can be verified effectively for every fixed vector of coefficients of $n+1$ variables.

Finally, what remains is to generate systematically the polynomials in $\mathbb{Q}[x_1, \ldots, x_{n+1}]$ and find p satisfying the previous condition. The search for p might be carried out as follows. We use the following notation: given a polynomial

$$p(x_1, \ldots, x_{n+1}) = \sum_{i=(i_1,\ldots,i_{n+1})} T_i x_1^{i_1} \cdots x_{n+1}^{i_{n+1}} \in \mathbb{Q}[x_1, \ldots, x_{n+1}]$$

and $M \in \mathbb{N}$, we will say that the weight of p, w_p, is less than or equal to the constant M if $\deg(p), |T_i| \leq M$.

It is clear then, that for all $M \in \mathbb{N}$ the set $\{p \in \mathbb{Q}[x_1, \ldots, x_{n+1}] \,|\, \mathrm{w}_p \leq M\}$ is a finite set.

Now, the search for p will proceed as follows: at stage k, the algorithm generates systematically the finitely many polynomials $p \in \mathbb{Q}[x_1, \ldots, x_{n+1}]$ of weight at most 2^k and checks whether the polynomial p satisfies the required condition. As we have already mentioned, this can be done by virtue of Tarski- Seidenberg. This procedure eventually halts and finds the desired polynomial because of the already proven existence.

$\square$

5.5 Conclusions

Putting together the previous argument, we obtain the following result illustrating the intimate connection between superperfect groups and homology spheres.

Theorem 5.5.1 *Let us assume $n \geq 5$. Given an effectively constructed sequence of superperfect groups $\{\widetilde{G_i}\}$ we can effectively construct a sequence of compact non-singular algebraic hypersurfaces $S_i \subset \mathbb{R}^{n+1}$, satisfying the following conditions:*

(i) *The S_i are homology spheres.*

(ii) *For all $i \geq 1$ $\pi_1(S_i) = \widetilde{G_i}$.*

We finally arrive to the main result of this monograph, dealing with the unrecognazibility of the n-spheres for $n \geq 5$. As we mentioned before, the statement of the next theorem, along with its proof, follows the exposition that appeared in the appendix of (Nab95).

Theorem 5.5.2 *For any $n \geq 5$ there is no algorithm which for a given polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$ whose zero set $Z(p)$ is a non-singular algebraic hypersurface decides whether or not $Z(p)$ is diffeomorphic to the sphere $\mathbb{S}^n$.*

Proof. Let us suppose that exists such procedure. From Theorem 4.4.2 there exists a Novikov sequence $\{\widetilde{G_i}\}$ whose elements are finite presentations of superperfect groups. Now it follows from the previous theorem that we can generate effectively a sequence of non-singular hypersurfaces $S_i \subset \mathbb{R}^{n+1}$. We also know that these S_i are homology spheres obtained as zero sets of polynomials $p \in \mathbb{Q}[x_1, \ldots, x_{n+1}]$. Even more, we also have that $\pi_1(S_i) = \widetilde{G_i}$, for $i \geq 1$. Now if we apply our procedure to the elements of the sequence $\{S_i\}$ we could decide which S_i are diffeomorphic to the n-sphere. But from Corollary 2.2.16 that means that we could solve the Triviality Problem for the Novikov sequence $\{\widetilde{G_i}\}$, which is impossible.

■

We conclude by showing that Novikov's theorem implies, for $n \geq 5$, the unrecognizability of all compact n-dimensional manifolds. See (CL06, p. 332).

Theorem 5.5.3 *Given any compact manifold M_0 of dimension $n \geq 5$, there is no algorithm that recognize M_0 among the class of all compact n-dimensional manifolds.*

Proof. Suppose for simplicity that M_0 is a connected n-dimensional manifold (possibly with a boundary or non-compact), which can be effectively recognized among the class of all compact n-dimensional manifolds. We will show that in this case it would be possible to recognize the n-dimensional sphere $\mathbb{S}^n$, which would contradict the Theorem 5.5.2. Let M be a compact n-dimensional manifold effectively generated from a Novikov sequence of groups. Let M_1 be its connected sum with M_0, i.e., $M_1 = M_0 \# M$. Now, apply our procedure to recognize M_0 to M_1. If the answer is *No*, it is clear that M is not a sphere. If the answer is *Yes*, note that the fundamental group of M is the trivial group. Indeed, the fundamental group of M_1 is a free product of the fundamental groups of M and M_0, at the same time it must coincide with the fundamental group of M_0. This is possible only if M is simply connected because the rank of a free product of two groups is equal to the sum of the ranks of the two free factors (this is a consequence of Grushko's theorem, see for example (Rot94, p. 393)). But the only simply connected n-dimensional manifold generated from a Novikov sequence of groups is the n-sphere. Thus, the recognizability of M_0 implies the recognizability of the sphere, which is impossible.

■

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